ELSEVIER
Journal of Pure and Applied Algebra 173 (2002) 87–120
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Tropological systems are points of quantales

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Received 3 March 2001; received in revised form 4 October 2001
Communicated by Y. Adámek

Abstract

We address two areas in which quantales have been used. One is of a topological nature, whereby quantales or involutive quantales are seen as generalized noncommutative spaces, and its main purpose so far has been to investigate the spectrum of noncommutative C*-algebras. The other sees quantales as algebras of abstract experiments on physical or computational systems, and has been applied to the study of the semantics of concurrent systems. We investigate connections between the two areas, in particular showing that concurrent systems, in the form of either set-theoretic or localic tropological systems, can be identified with points of quantales by means of a suitable adjunction, which indeed holds for a much larger class of so-called "tropological models". We show that in the case of tropological models in factor quantales, which still generalize tropological systems, the identification of models and (generalized) points preserves all the information needed for describing the observable behaviour of systems. We also define a notion of morphism of models that generalizes previous definitions of morphism of systems, and show that morphisms, too, can be defined in terms of either side of the adjunction, in fact giving us isomorphisms of categories. The relation between completeness notions for tropological systems and spatiality for quantales is also addressed, and a preliminary partial preservation result is obtained. © 2001 Elsevier Science B.V. All rights reserved.

MSC: Primary 06F07; secondary 18B20; 18F99; 46L89; 54A05; 68Q85

1. Introduction

It has long been argued that the space of irreducible representations of a noncommutative C*-algebra is not adequately handled by conventional topology, and quantales [16] are meant to remedy this by providing a notion of spectrum for C*-algebras

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PII: S0022-4049(01)00156-6

that generalizes the localic spectrum of the commutative case, yielding an algebraic framework that places the insights of [9,2,3] into a lattice theoretic kind of noncommutative topology. The spectrum of a unital C*-algebra A is defined [17] to be the involutive quantale $\mathrm{Max}\,A$ of closed subspaces of A (see also [19,18]), which is motivated by a noncommutative generalization of the constructive spectrum of commutative C*-algebras that was presented in [5,6] using propositional geometric logic. Quantales are a generalization of locales, and corresponding generalizations of the notion of point of a locale have been studied in [24,19] in the case of involutive quantales, and in [22,14] for arbitrary quantales (see also the survey [23]). The irreducible representations of a C*-algebra A determine irreducible representations of $\mathrm{Max}\,A$ [19]. These are examples of points of $\mathrm{Max}\,A$, and in [19] such points are given a nice algebraic characterization.

In the present paper we mimic the above ideas, but in an entirely different field, the aim being to assess the extent to which quantales are also capable of describing "spaces" whose points are instances of a particular kind of dynamical system found in computer science, namely in concurrency theory; the definition of such dynamical systems uses quantales in a natural way [1,29,28], and our intention is to relate these rather different applications of quantales. More precisely, we will see that the examples in concurrency can be recast into a form similar to that of [19], whereby spectra are described in the category of unital quantales, and in such a way that systems themselves can be identified with certain points. Such an example is also interesting because it adds credit to the idea that noncommutative topology is related to intrinsically dynamical notions of space, thus reinforcing a similar intuition that often appears in noncommutative geometry [8], for instance when studying examples such as spaces of leaves of foliations, spaces of irreducible unitary representations of discrete groups, etc. However, we stress that in the present paper the connection of quantales to C*-algebras is being used essentially as motivation, since no results from C*-algebra theory will be used. Hence, we are mostly concerned with displaying examples of "noncommutative spaces", with respect to which C*-algebras are, at least for now, mainly related by analogy. In [15] further connections between quantales and C*-algebras are studied.

1.1. Concurrency and tropological systems

In concurrency, *labelled transition systems* (LTSs) are models of (interleaving) concurrent systems; each LTS consists of a set P of *states*, or *processes*, and a map $\xrightarrow{-} : Act \to \mathscr{P}(P \times P)$, where Act is the set of *actions* and for each action $\alpha \in Act$ the relation $\xrightarrow{\alpha} \subseteq P \times P$ is the *transition relation* of α; $p \xrightarrow{\alpha} q$ means that if the system is at the state p then α can be performed, and that by doing so the resulting state can be q. The problem with such models is that they provide very little information about the semantics of concurrent processes, and additional *behavioural equivalences* have to be supplied [10,11], whereby certain states are considered to be equivalent in the sense that they have the same *observable behaviour*.

It is commonly assumed that such equivalences rely on notions of *experimental observation*, and in [1] this was made explicit by taking the actions to be some of the generators of a unital quantale (i.e., a monoid in the category of sup-lattices **SL** [13]—see Section 2), which however may have other generators. In other words, the quantale is an algebra of *finite run-time observations*; performing actions from Act is a way of observing the system, but there may be other ways, which correspond to other quantale generators (e.g., trying to press a key but noticing that it is disabled, during which the state of the system is not changed, or seeing that an action is possible by observing it on a menu, without actually doing it, etc.). The multiplication of the quantale is then understood to be a usually noncommutative and nonidempotent conjunction: $a \cdot b$ means "a and then b" (in fact this idea was originally present in [16], and can also be found in [34]), and the transition relations are extended to all the observations so as to yield a unital quantale homomorphism $\xrightarrow{-} : Q \to \mathscr{P}(P \times P)$ (where $\mathscr{P}(P \times P)$ is a quantale under the inclusion ordering, with multiplication given by composition of relations and unit being the diagonal relation), i.e., so that the following conditions hold for all $p, q \in P$, $a, b \in Q$, and $X \subseteq Q$:

- $p \xrightarrow{e} q$ if and only if $p = q$,
- $p \xrightarrow{a \cdot b} q$ if and only if $p \xrightarrow{a} r \xrightarrow{b} q$ for some $r \in P$,
- $p \xrightarrow{\bigvee X} q$ if and only if $p \xrightarrow{a} q$ for some $a \in X$.

The order in the quantale thus tells us that if $a \leqslant b$ and $p \xrightarrow{a} q$ then $p \xrightarrow{b} q$; we can say that a is a *particular way* of observing b. There is another order, however, that does not take into account the states after performing the observations, namely we write $a \leqslant' b$ if for all states p, if $p \xrightarrow{a} q$ *for some* q then $p \xrightarrow{b} r$ *for some* r. In this way we are seeing Q as a set of *capabilities* of processes, rather than a set of observations—if p *can* do a then it *can* do b; the algebraic structure in both cases is different, for the quotient $Q/(\leqslant' \cap \geqslant')$ is no longer a quantale but only a left Q-module (i.e., a left Q-action in **SL**). This leads to the idea that the capabilities of processes should in general be contained in a left Q-module L (in [30] there are also examples in which L is not a quotient of Q). The module L can be thought of as a "topology" on P; to be more precise, we define a map $\Pi : L \to \mathscr{P}(P)$ with the following properties, for all $p \in P$, $Y \subseteq L$, $a \in Q$, and $\varphi \in L$, where we write $p \models \varphi$ for $p \in \Pi(\varphi)$ and $\top_L$ for the top of L:

- $p \models \top_L$,
- $p \models \bigvee Y$ if and only if $p \models \varphi$ for some $\varphi \in Y$,
- $p \models a \cdot \varphi$ if and only if $p \xrightarrow{a} q$ and $q \models \varphi$ for some $q \in P$.

The first two conditions tell us that Π is a sup-lattice homomorphism which is strong (i.e., it preserves the top) and are what we would expect if L were a frame and Π a frame homomorphism, in which case the image set $\Pi[L] \subseteq \mathscr{P}(P)$ would indeed be a topology on P; the third condition replaces preservation of meets and jointly with the second condition tells us that Π is homomorphism of left Q-modules, where $\mathscr{P}(P)$ is a left $\mathscr{P}(P \times P)$-module with action given by inverse image, and so also a Q-module via the change of quantale induced by the map $\xrightarrow{-} : Q \to \mathscr{P}(P \times P)$. The structure

$(P, Q, \overset{-}{\rightarrow}, L, \Pi)$ provides a generalization of the notion of topological system of [33] (frames are replaced by pairs (Q, L), and the satisfaction relation splits into $\overset{-}{\rightarrow}$ and Π), and is called a *tropological system* [29,28], from the greek *trópos*, change, as opposed to *tópos*, place—see also [30] for a localic generalization of this notion in a constructive setting.

Now we can define a "specialization preorder" $\lesssim$ on P, called the *behavioural preorder*, by, for all states p and q,

$$p \lesssim q \quad \Leftrightarrow \quad \forall_{\varphi \in L}(p \models \varphi \Rightarrow q \models \varphi).$$

Hence, $p \lesssim q$ means that q has all the capabilities that p has. In the applications to process semantics it is typically the case that for each pair (Q, L) with $Act \subseteq Q$, each LTS over Act can be extended in a unique way to a tropological system; that is, there is a unique tropological system $(P, Q, \overset{-}{\rightarrow}, L, \models)$, the *tropological extension* of the LTS, whose transition relation coincides with that of the LTS for all actions $\alpha \in Act$ [29,28,30]. Hence, the pair (Q, L) automatically induces a behavioural equivalence on P, namely that which is associated to the tropological extension, and thus we obtain a process semantics via quantales and modules. In practice this allows us to adopt the point of view, as we do in the present paper, that a concurrent system (at least of the interleaving kind) *is* a tropological system.

Another aspect that has some relevance for tropological systems is that they can be related by means of suitable notions of morphism. There is more than one such notion, in fact. The morphisms of [29] served the purpose of defining implementations of systems on other systems, whereas in [30] morphisms are required to preserve more structure. In particular, they are homomorphisms of quantale modules. Also, in [30] a more general notion of tropological system is addressed, the set of states being replaced by a locale (see also Section 3.3), which gives rise to categories of systems with final objects that are useful for defining notions of final semantics.

1.2. Tropological systems as points

At this stage one may ask whether the use of quantales just described may bear any resemblance to the original application of quantales to C*-algebras. As stated in the beginning, our aim is to convey the idea that just as involutive quantales are capable of modelling the spectrum of a C*-algebra, which is a noncommutative space, quantales can also serve the purpose of describing "spaces" of concurrent systems, which may thus acquire the status of noncommutative spaces in some sense.

Indeed we will see that this is so, at least in the sense that tropological systems for a pair (Q, L) can be identified with points of a suitable quantale. Notice that on the one hand this is to be expected because any tropological system for (Q, L) includes a quantale homomorphism $\overset{-}{\rightarrow} : Q \to \mathscr{P}(P \times P)$ which is basically a representation of Q on $\mathscr{P}(P)$ (see Section 2.1). However, this does not tell us anything about the role of L, and furthermore $\overset{-}{\rightarrow}$ is not required to preserve the top, whereas points of quantales are irreducible representations (see Section 2.2). So our strategy will be to obtain, from each pair (Q, L), another quantale, which is presented by generators and relations and which by analogy with C^*-algebras we think of as being the "spectrum" of (Q,L), here denoted by $S(Q,L)$. In fact we shall also include a right Q-module R as an algebra of "time reversed capabilities", essentially because there is no technical burden in doing so, and furthermore such entities can be justified when we make assertions about the past (e.g., "a was just done", rather than "a can be done"—for instance, such situations arise in quantum mechanics, where systems are *prepared* in certain ways before we perform experiments on them, and such preparations can be seen as reversed capabilities). We thus obtain a generalized notion of tropological system, and a spectrum $S(R,Q,L)$ for each triple (R,Q,L). Briefly, the main goal of this paper is to show that tropological systems for a triple (R,Q,L) can be identified with points of the spectrum $S(R,Q,L)$, and furthermore in such a way that the "tropological information" present in (R,Q,L)—in particular the behavioural equivalence relations and the morphisms of tropological systems—can be completely recovered from the spectrum. We also present preliminary results concerning the extent to which the spectrum of a triple should be expected to be "spatial".

In more detail, we show in Section 3.2 that a certain class of irreducible representations of $S(R,Q,L)$ on powerset sup-lattices $\mathscr{P}(P)$ correspond bijectively with tropological systems for (R,Q,L) with set of states P. In fact, what we do is more general, namely we obtain an adjunction that gives us a correspondence for homomorphisms much more general than representations, referred to as *tropological models*. Furthermore we address an even more general notion of tropological system, in which it is possible to have $p \xrightarrow{e} q$ with $p \neq q$, provided that $p \xrightarrow{e} p$ for all states p. Such a generalization is intended to apply to systems which are capable of performing internal activity hidden from the observer [26,28], and has the advantage of making the theory closer to that of [19], whose quantale homomorphisms are pre-unital in the sense that the quantale unit is mapped above the unit. In Section 3.3 the role of pre-unital homomorphisms is studied and compared to that in [19], via a notion of unital spectrum which is also applied to the localic tropological systems of [30].

In Section 4 we study a smaller class of tropological models, based on the notion of factor quantale of [22]. We show, in Section 4.1, that the "topologies" induced by L (and also R) are preserved in the translation from systems to points in the case of factor models, which as a consequence shows that the behavioural pre-orders of tropological systems can be recovered from the translations of these into points. A word of caution is in order: whereas in [19] the spectra of C^*-algebras are involutive quantales, we mostly ignore involution in the present paper, but in Sections 3.2 and 4.1 we discuss this matter briefly. In Section 4.2 we study categories of tropological models, which are generalizations of the categories of tropological systems of [30]. We extend the results of Section 3.2 in the case of factor models, showing that not only the objects of these categories are "preserved" when moving from systems to points, but also the morphisms are, and in fact we obtain isomorphisms of categories. The categories of models that we define are also quite general, being based on a notion of morphism between Galois connections which in particular subsumes sup-lattice homomorphisms and continuous maps of topological spaces [25]. Finally, in Section 4.3 we address spatiality of quantales and completeness in tropological systems, mainly with the purpose of providing a preliminary account of problems still to be solved (Theorem 4.12).

2. Preliminaries

In this section we provide technical preliminaries and some general background about quantales, modules, and tropological systems, and fix notation that will be used in later sections.

We shall denote the category of sup-lattices by **SL** (complete lattices with homomorphisms being the maps that preserve joins). We recall that **SL** is a closed monoidal category with biproducts [13]. We denote the biproduct of two sup-lattices L and M by $L \oplus M$, and write $L \amalg M$ for their disjoint union as sets. The tensor product of L and M is denoted by $L \otimes M$, and similarly to vector spaces it is the image of a universal bimorphism, where a sup-lattice bimorphism $f : L \times M \to N$ is a map that preserves joins in each variable separately: $f(\bigvee X, y) = \bigvee\{f(x,y) \mid x \in X\}$ and $f(x, \bigvee Y) = \bigvee\{f(x,y) \mid y \in Y\}$. Concretely, $L \otimes M$ can be identified with the set of those subsets $I \subseteq L \times M$ such that $(x, \bigvee Y) \in I \Leftrightarrow \{x\} \times Y \subseteq I$ and $(\bigvee X, y) \in I \Leftrightarrow X \times \{y\} \subseteq I$ for all $x \in L$, $y \in M$, $X \subseteq L$, and $Y \subseteq M$. The universal bimorphism $L \times M \to L \otimes M$ is defined by $(x,y) \mapsto x \otimes y$, where $x \otimes y$ is the least such set I that contains the pair (x,y); that is, $x \otimes y = \downarrow(x,y) = \{(z,w) \in L \times M \mid z \leqslant x,\ w \leqslant y\}$.

We denote the top of a sup-lattice L by $\top_L$, or $\top$, and the bottom by 0_L or 0. The two-element sup-lattice $\{0, \top\}$ is denoted by 2. A homomorphism of sup-lattices $f : L \to M$ is said to be *strong* if it preserves the top: $f(\top_L) = \top_M$.

Let $f : L \to M$ be a sup-lattice homomorphism. Its right adjoint $f_* : M \to L$ preserves meets and thus defines another sup-lattice homomorphism $\hat{f} : M^{\mathrm{op}} \to L^{\mathrm{op}}$, called the *dual* of f.

Quotients of sup-lattices can be conveniently handled by means of closure operators. Let L be a sup-lattice and j a closure operator on L. The set of j-closed elements $L_j = \{x \in Q \mid x = j(x)\}$ is a sup-lattice with joins $\bigvee^j(x_i) = j(\bigvee x_i)$, and the map $j : L \to L_j$ is a (surjective) homomorphism of sup-lattices. Furthermore, every sup-lattice quotient arises in this way up to isomorphism.

For further facts about sup-lattices we refer to [13].

2.1. Quantales

A *quantale* [16] (see also [31,23]) is a sup-lattice equipped with an associative *multiplication* $\cdot$ that distributes over arbitrary joins,

$$a \cdot \left(\bigvee_i b_i\right) = \bigvee_i (a \cdot b_i), \qquad \left(\bigvee_i a_i\right) \cdot b = \bigvee_i (a_i \cdot b),$$

i.e., a quantale is a semigroup in **SL**. A quantale Q is *unital* if the multiplication has a unit, which we denote by e_Q, or simply e.

A *homomorphism* of quantales $h : Q \to Q'$ is a function that preserves the multiplication and arbitrary joins. In the case that Q and Q' are both unital, we say that h is *pre-unital* if $h(e_Q) \geqslant e_{Q'}$, and *unital* if $h(e_Q) = e_{Q'}$ (Mulvey and Pelletier [19] define unital as being what we call pre-unital).

Following [20], given a sup-lattice L we denote by $\mathcal{Q}(L)$ the unital quantale of sup-lattice endomorphisms of L, whose order is computed pointwise, and whose multiplication is composition: $f \cdot g = f; g = g \circ f$ (in other papers, such as [23,14], the convention is $f \cdot g = f \circ g$). The unit of $\mathscr{Q}(L)$ is the identity map 1_L. A particular example of such a quantale occurs when $L = \mathscr{P}(P)$ for some set P, and we have a unital isomorphism of quantales $\mathscr{Q}(\mathscr{P}(P)) \cong \mathscr{P}(P \times P)$ (each endomorphism f of $\mathscr{P}(P)$ is mapped to the relation θ such that $p\theta q$ if and only if $q \in f(p)$ [20]).

An element of a quantale $a \in Q$ is *right-sided* if $a \cdot \top \leqslant a$ (equivalently, $a \cdot \top = a$ if Q is unital), and *left-sided* if $\top \cdot a \leqslant a$. The set of right-sided elements is denoted by $\mathrm{R}(Q)$, and the set of left-sided elements is denoted by $\mathrm{L}(Q)$. Both $\mathrm{R}(Q)$ and $\mathrm{L}(Q)$ are subquantales of Q (but not unital subquantales). We have order isomorphisms $\mathrm{L}(\mathscr{Q}(S)) \cong S$ and $\mathrm{R}(\mathscr{Q}(S)) \cong S^{\mathrm{op}}$ [20], and thus $\mathrm{L}(\mathscr{P}(P \times P)) \cong \mathrm{R}(\mathscr{P}(P \times P)) \cong \mathscr{P}(P)$.

A (*unital*, *pre-unital*) *representation* of Q on a sup-lattice L is a (unital, pre-unital) homomorphism $r : Q \to \mathscr{Q}(L)$. Following [19], if r is strong we say that the representation is *irreducible*.

An *involutive quantale* Q is a quantale equipped with an *involution* $(-)^* : Q \to Q$, i.e., a join preserving operation that makes Q an involutive semigroup: $(a \cdot b)^* = b^* \cdot a^*$. Any involutive quantale Q satisfies $\top^* = \top$ and, if Q is unital, $e^* = e$. An important example is that $\mathscr{Q}(S)$ is an involutive quantale whenever S is equipped with a *duality*, i.e., a dual automorphism $(-)' : S \to S^{\mathrm{op}}$ [20]; the involution is then given by

$$f^*(y) = \left(\bigvee \{x \in S \mid f(x) \leqslant y'\}\right)'.$$

We will usually refer to such a sup-lattice S just as a *self-dual* sup-lattice, with the understanding that the involution on $\mathscr{Q}(S)$ is determined by some pre-specified duality on S in the manner indicated above. As an example, the quantale $\mathscr{Q}(\mathscr{P}(P))$ has an involution determined by the complement on $\mathscr{P}(P)$; hence, the isomorphism $\mathscr{Q}(\mathscr{P}(P)) \cong \mathscr{P}(P \times P)$ defines an involution on $\mathscr{P}(P \times P)$, and this coincides with reversal: $R^* = \{(q,p) \in P \times P \mid (p,q) \in R\}$. All the above definitions generalize in the obvious way to involutive quantales, e.g., an *involutive homomorphism* is a quantale homomorphism that also preserves the involution, etc.

We define the following categories:

- $\mathbf{Qu}$, whose objects are the unital quantales and whose arrows are the pre-unital homomorphisms;
- $\mathbf{Qu}_e$, with the same objects, but restricted to unital homomorphisms.

The subcategories of the above that result from restricting to involutive quantales and involutive homomorphisms are denoted by $\mathbf{Qu}^*$ and $\mathbf{Qu}_e^*$, respectively, and given each category of quantales C, the subcategory that restricts to strong homomorphisms is denoted by $\bar{C}$: $\overline{\mathbf{Qu}}$, $\overline{\mathbf{Qu}}_e$, $\overline{\mathbf{Qu}}^*$, and $\overline{\mathbf{Qu}}_e^*$.

2.2. Points and spatiality

There are several notions of point for quantales and involutive quantales, all meant to generalize the notion of point that exists in the theory of locales. The first of these was put forward in [32] for idempotent right-sided quantales, and it is based on a generalization of the notion of prime element of a frame. However, in this paper we are concerned with points for more general quantales, which we now address.

For the general case of an arbitrary involutive quantale, the concept of *simple involutive quantale* was introduced in [24], corresponding to an involutive quantale Q with the property that any surjective involutive homomorphism from Q is either zero or an isomorphism. Simple involutive quantales thus correspond to the geometric notion of singleton space, for in a dual category they can be thought of as those "spaces" the inclusions into which are either empty or isomorphisms. The points of an involutive quantale Q can then be defined to be the strong involutive homomorphisms $p: Q \to Q'$, where Q' is an arbitrary simple quantale, indeed providing a generalization of the notion of point for locales. In [24] it is further shown that any involutive quantale $\mathscr{Q}(S)$ with S self-dual is simple, and thus involutive irreducible representations on self-dual sup-lattices are examples of points. Moreover, with respect to a natural generalization of the notion of spatiality of locales (Q is *spatial* if for all $a, b \in Q$ the equality $a = b$ holds if for all points p of Q we have $p(a) = p(b)$), it suffices to consider points that are irreducible representations; that is, an involutive quantale is spatial if and only if for all $a, b \in Q$ the equality $a = b$ holds if for all involutive irreducible representations $r: Q \to \mathscr{Q}(S)$ on self-dual sup-lattices S we have $r(a) = r(b)$ [24, Theorem 5.8].

The definitions and results of [24] were adapted in [22] to noninvolutive quantales, which in various aspects requires only that involution be forgotten. For instance, simple noninvolutive quantales are those from which any surjective homomorphism is either zero or an isomorphism. Points are strong homomorphisms into simple quantales, and a quantale is spatial in the natural way if and only if all its elements can be distinguished by irreducible representations [22]. Hence, from the point of view of this notion of spatiality it suffices again to consider points to be irreducible representations.

In [14] points of quantales, defined as certain *prime elements*, were studied and shown to correspond in a precise way to irreducible representations, yielding again a generalization of the notions of point and spatiality that exist for locales—a quantale Q is spatial if and only if every $a \in Q$ is a meet of primes—which adds more robustness to the theory of points and spatiality started in [24] (see also the survey [23]).

In [19] the approach is somewhat different, as points of quantales not only generalize the notion of point of locale theory but they are also required to match the notion of irreducible representation of the theory of C*-algebras. In fact, this leads to a refinement of the previous notions based on simple quantales, for a point of an involutive quantale Q is defined to be an *algebraically irreducible involutive representation* $p: Q \to \mathscr{Q}(S)$, where S is an atomic orthocomplemented sup-lattice. We shall not give the exact definition of "algebraically irreducible" (which implies irreducible) here, but we remark that the reason behind it is that these points are meant to correspond bijectively, in the case of the spectrum $\operatorname{Max} A$ of a C*-algebra, to the equivalence classes of irreducible representations of A [19].

It is worth remarking that in spite of the robustness and aesthetic appeal of several of the above results, there is still not a definitive theory of points and spatiality, and in fact it is not clear that there should be a single generalization of the notion of spatiality of locales which is suitable for all purposes. For instance, an example by Kruml in [23] shows that quantales of the form $\operatorname{Max} A$ should usually be expected *not* to be spatial in the sense described above (see also [15]), and an alternative notion of spatiality, which also generalizes that of locales and according to which $\mathrm{Max}\,A$ is spatial, was put forward in [18].

2.3. Nuclei and quotients

We begin by recalling a few facts about quotients of quantales, most of which can be found in [31].

A (*quantic*) *nucleus* [21,31] on a quantale Q is a closure operator $j: Q \to Q$ that satisfies, for all $x, y \in Q$,

$$j(x) \cdot j(y) \leqslant j(x \cdot y).$$

The set of j-closed elements $Q_j = \{x \in Q \mid x = j(x)\}$ is a quantale with joins $\bigvee^j(x_i) = j(\bigvee x_i)$ and multiplication $x \cdot_j y = j(x \cdot y)$, it is unital if Q is, with unit $j(e)$, and the map $j: Q \to Q_j$ is a (surjective) homomorphism of quantales, unital if Q is unital. Furthermore, every quantale quotient arises in this way up to isomorphism.

Given a nucleus j on a quantale Q, the set Q_j is closed under meets and under the left and right residuations $\to_{\mathrm{l}}$ and $\to_{\mathrm{r}}$ which are right adjoints to multiplication,

$$a \cdot c \leqslant b \quad \Leftrightarrow \quad c \leqslant a \to_{\mathrm{r}} b,$$

$$c \cdot a \leqslant b \quad \Leftrightarrow \quad c \leqslant a \to_{\mathrm{l}} b,$$

where being closed under residuation means that for all $a \in Q_j$ and $b \in Q$ we have $b \to_{\mathrm{r}} a \in Q_j$ and $b \to_{\mathrm{l}} a \in Q_j$. Conversely, any subset $Q' \subseteq Q$ closed under meets and the residuations is of the form Q_j for the nucleus j defined by $j(x) = \bigwedge\{y \in Q' \mid x \leqslant y\}$.

The set $\mathrm{N}(Q)$ of nuclei on a quantale Q is a complete lattice under the pointwise order, with meets being calculated pointwise: $j \leqslant k \Leftrightarrow \forall_{x \in Q}\,(j(x) \leqslant k(x))$, and $(\bigwedge_\alpha j_\alpha)(x) = \bigwedge_\alpha (j_\alpha(x))$. Furthermore, we have $j \leqslant k \Leftrightarrow Q_k \subseteq Q_j$, and the join of nuclei corresponds to intersection of the respective sets of closed elements: $j = \bigvee_\alpha j_\alpha$ if and only if $Q_j = \bigcap_\alpha Q_{j_\alpha}$.

Let Q be a unital quantale, and R a subset of $Q \times Q$. It is easy to see that there is a least quantic nucleus j such that $j(y) = j(z)$ for all $(y, z) \in R$, given explicitly by $j_R = \bigwedge\{j \in \mathrm{N}(Q) \mid j(y) = j(z) \text{ for all } (y, z) \in R\}$ (in other words, Q_{j_R} is isomorphic to the quotient of Q by the quantale congruence relation generated by R, but we prefer to work with nuclei instead of congruence relations). The following proposition is not in [31] and provides a useful characterization of the set Q_{j_R}, analogous to that of [13, Proposition I.4.3] for sup-lattices. We address only unital quantales, which are the ones that interest us in this paper.

Proposition 2.1. *Let Q be a unital quantale, and $R \subseteq Q \times Q$ a set. Then Q_{j_R} coincides with the set Q_R of those elements $a \in Q$ such that for all $(y, z) \in R$ and all $b, c \in Q$ we have $b \cdot y \cdot c \leqslant a \Leftrightarrow b \cdot z \cdot c \leqslant a$.*

Proof. First, it is easy to see that Q_R is closed under meets and residuations, and thus it defines a nucleus k_R on Q. Furthermore, for all $(y, z) \in R$ and $a \in Q$ we have $y \leqslant k_R(a)$

if and only if $z \leqslant k_R(a)$, and thus from $y \leqslant k_R(y)$ and $z \leqslant k_R(z)$ we conclude $z \leqslant k_R(y)$ and $y \leqslant k_R(z)$, i.e., $k_R(y) = k_R(z)$. Hence, $j_R \leqslant k_R$, i.e., $Q_R \subseteq Q_{j_R}$. Conversely, let us prove that $Q_{j_R} \subseteq Q_R$. Let $a, b, c \in Q$ and $(y, z) \in R$. Then,

$$\begin{aligned} b \cdot y \cdot c \leqslant j_R(a) \;&\Leftrightarrow\; j_R(b \cdot y \cdot c) \leqslant j_R(a) \Rightarrow j_R(b) \cdot j_R(y) \cdot j_R(c) \leqslant j_R(a) \\ &\Leftrightarrow j_R(b) \cdot j_R(z) \cdot j_R(c) \leqslant j_R(a) \Rightarrow b \cdot z \cdot c \leqslant j_R(a). \end{aligned}$$

In a similar way we conclude that $b \cdot z \cdot c \leqslant j_R(a) \Rightarrow b \cdot y \cdot c \leqslant j_R(a)$, and thus $j_R(a) \in Q_R$. $\square$

2.4. Generators and relations

Let G be a set (of "generators"). The unital quantale freely generated by G is $\mathscr{P}(G^*)$, where G^* is the free monoid generated by G (i.e., the monoid of finite sequences of symbols from G with multiplication given by concatenation). The order in $\mathscr{P}(G^*)$ is inclusion, and multiplication is given by pointwise concatenation: $X \cdot Y = \{st \mid s \in X, t \in Y\}$. This leads to presentations by generators and relations, as follows.

Definition 2.2. Let G and $R \subseteq G^* \times G^*$ be sets. The unital quantale *presented* by the generators in G and the relations in R is $\mathbf{Qu}_e\langle G \mid R\rangle \stackrel{\text{def}}{=} \mathscr{P}(G^*)_R$. The *injection of generators* $[-] : G \to \mathbf{Qu}_e\langle G \mid R\rangle$ is the map defined by $[x] = j_R(\{x\})$ for each generator $x \in G$.

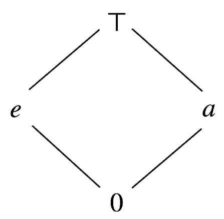
Proposition 2.3.

$$\mathbf{Qu}_e\langle G \mid R\rangle = \{X \subseteq G^* \mid \forall_{(Y,Z)\in R} \forall_{s,t\in G^*} (sYt \subseteq X \;\;\Leftrightarrow\;\; sZt \subseteq X)\}$$
$$[sYt \stackrel{\text{def}}{=} \{syt \mid y \in Y\}, \text{etc.}].$$

Proof. Easy consequence of Proposition 2.1. $\square$

This can also be derived from the results about presentations by generators and relations of [1], which are expressed in terms of coverage relations in a similar way to the coverage relations for frames in [12].

Example 2.4. $\mathbf{Qu}_e\langle\{\alpha\} \mid \{(\{\alpha\alpha\}, \{\alpha\})\}\rangle$ is isomorphic to the four-element quantale whose order and multiplication table are as follows:



$x \cdot y$	0	e	a	$\top$
0	0	0	0	0
e	0	e	a	$\top$
a	0	a	a	a
$\top$	0	$\top$	a	$\top$

Concretely, in terms of subsets of $\{\alpha\}^*$ we have the following identifications, where ε denotes the empty sequence, and $\{\alpha\}^+$ is the set of nonempty sequences of α's:

$$e = \{\varepsilon\},$$

$$a = \{\alpha\}^+,$$

$$\top = \{\alpha\}^*,$$

$$0 = \emptyset.$$

The injection of generators $[-]: G \to \mathbf{Qu}_e\langle G \mid R\rangle$ is a universal map amongst those maps $f: G \to Q$ (with Q a unital quantale) that respect the relations in R, by which we mean that their homomorphic extensions $\bar{f}: \mathscr{P}(G^*) \to Q$ satisfy $\bar{f}(Y) = \bar{f}(Z)$ for each $(Y,Z) \in R$. This fact allows us to adopt a notation that is easier to read and which consists of replacing the relations by those conditions with respect to which the injection of generators is universal (we also drop "{" and "}" where possible). For instance, the above example would read $\mathbf{Qu}_e\langle \alpha \mid [\alpha]\cdot[\alpha]=[\alpha]\rangle$, and, in general, a relation $(\{x_1^{(i)} \ldots x_{n_i}^{(i)} \mid i \in I\}, \{y_1^{(j)} \ldots y_{n_j}^{(j)} \mid j \in J\})$ is replaced by the condition $\bigvee_{i\in I} [x_1^{(i)}] \cdot \ldots \cdot [x_{n_i}^{(i)}] = \bigvee_{j \in J} [y_1^{(j)}] \cdot \ldots \cdot [y_{n_j}^{(j)}]$.

Example 2.5. Every unital quantale Q is a quotient of a free unital quantale, for the homomorphic extension $1_Q^\sharp: \mathscr{P}(Q^*) \to Q$ of the identity $1_Q: Q \to Q$ is a surjective unital homomorphism. In terms of generators and relations this means that Q is isomorphic to $\mathbf{Qu}_e\langle Q \mid R\rangle$, where R is the set of defining relations of the form

$$\bigvee_i [x_i] = \left[\bigvee_i x_i\right],$$

$$[x]\cdot[y] = [x \cdot y],$$

$$e = [e_Q],$$

standing, respectively, for pairs $(\bigcup_i \{x_i\}, \{\bigvee_i x_i\})$, $(\{xy\}, \{x \cdot y\})$, and $(\{\varepsilon\}, \{e_Q\})$.

Example 2.6. Let L be a sup-lattice. The quantale $\mathscr{T}(L) = \mathbf{Qu}_e\langle L \mid [\bigvee_i x_i] = \bigvee_i [x_i] (x_i \in L)\rangle$ (i.e., the quantale generated by L with joins being preserved in the presentation) is isomorphic to the *tensor quantale* $\bigoplus_{n\in\omega} L^{\otimes n}$, whose quantale structure is obtained in a similar way to the algebra structure of a tensor algebra over a vector space, and the injection of generators corresponds to the coprojection $L = L^{\otimes 1} \to \bigoplus_{n \in \omega} L^{\otimes n}$. Concretely, $\mathscr{T}(L)$ can be described as consisting of those subsets $I \subseteq L^*$ such that, for all $s,t \in L^*$ and $X \subseteq L$, $s(\bigvee X)t \in I$ if and only if $\{sxt \mid x \in X\} \subseteq I$.

We conclude this section remarking that involutive quantales can be constructed in a very similar way, based on a notion of *involutive nucleus* [i.e., a quantic nucleus that also satisfies $j(x)^* \leqslant j(x^*)$] and on the fact that the free involutive unital quantale on a set G equals $\mathscr{P}(M)$, where M is the free involutive monoid on G, which can be identified with $(G \amalg G)^*$. However, we shall mostly ignore involution in this paper and so we omit the details of this construction (in fact such constructions exist for arbitrary finitary algebraic theories on sup-lattices [26], of which another example are the quantale modules addressed below).

2.5. Quantale modules

Let Q be a unital quantale. A *left Q-module* is a sup-lattice L equipped with a left join-preserving *action* of Q on L (also denoted by "$\cdot$"); that is, for all $a, b, a_i \in Q$, $x, x_j \in L$,

$$a \cdot (b \cdot x) = (a \cdot b) \cdot x,$$

$$\left(\bigvee_i a_i\right) \cdot x = \bigvee_i (a_i \cdot x),$$

$$a \cdot \left(\bigvee_j x_j\right) = \bigvee_j (a \cdot x_j).$$

The module is *pre-unital* if $e \cdot x \geqslant x$ for all $x \in L$, *unital* if $e \cdot x = x$ for all $x \in L$, and *irreducible* if $\top_Q \cdot x = \top_L$ for all $x \neq 0_L$. A *right Q-module* is defined in the same way, with Q acting on the right. Typical examples of quantale modules are:

- If Q is a unital quantale, Q itself is both a left and a right unital module over itself, with action given by multiplication; $\mathrm{R}(Q)$ is a unital left Q-module with action given by multiplication on the left; $\mathrm{L}(Q)$ is a unital right Q-module, with action given by multiplication on the right.
- A unital right Q-module structure on L is equivalent to a unital representation $Q \to \mathscr{Q}(L)$, and thus L is a unital right module over $\mathscr{Q}(L)$; the action coincides with that of $\mathrm{L}(\mathscr{Q}(L))$ via the order isomorphism $L \cong \mathrm{L}(\mathscr{Q}(L))$. Similarly, a pre-unital right Q-module L corresponds to a pre-unital representation $Q \to \mathscr{Q}(L)$, and an irreducible right Q-module L corresponds to an irreducible representation $Q \to \mathscr{Q}(L)$.
- By sup-lattice duality [13] the quantale $\mathscr{Q}(L^{\mathrm{op}})$ is isomorphic to $\mathscr{Q}^*(L)$ (i.e., $\mathscr{Q}(L)$ with multiplication reversed), and thus L^{op} is a unital left module over $\mathscr{Q}(L)$; the action coincides with that of $\mathrm{R}(\mathscr{Q}(L))$ via the order isomorphism $L^{\mathrm{op}} \cong \mathrm{R}(\mathscr{Q}(L))$. It follows that the dual L^{op} of a left Q-module L is a right Q-module, and conversely. Also, the dual $\hat{f}$ of a left Q-module homomorphism is a right Q-module homomorphism, and conversely.
- Let P be a set. From the above examples it follows that $\mathscr{P}(P)$ is both a left and a right unital module over the quantale of binary relations $\mathscr{P}(P \times P)$. Explicitly, the right action is direct image, and the left action is inverse image: for $R \subseteq P \times P$ and $X \subseteq P$, $R \cdot X = \{y \in P \mid yRx \text{ for some } x \in X\}$ and $X \cdot R = \{y \in P \mid xRy \text{ for some } x \in X\}$.

Let Q be a unital quantale. A *homomorphism* of left Q-modules is a sup-lattice homomorphism f that commutes with the action: $f(a \cdot x) = a \cdot f(x)$, and similarly for right

Q-modules. The categories of left or right Q-modules have many properties that are similar to those of ring modules (e.g., they have biproducts), but we will restrict to strong homomorphisms, for which $\mathrm{R}(Q)$ is the initial unital left Q-module [29, Proposition 3.6(8)]: if L is another unital left Q-module the unique strong homomorphism $\mathrm{R}(Q) \to L$ is given by $a \cdot \top_Q \mapsto a \cdot \top_L$. In fact, this also holds if L is pre-unital because the submodule $Q \cdot \top_L \subseteq L$ is always unital. Hence, we denote the categories of pre-unital left and right Q-modules with strong homomorphisms, respectively, by ${}_Q\overline{\mathbf{Mod}}$ and $\overline{\mathbf{Mod}}_Q$, and obtain the following proposition.

Proposition 2.7. *Let Q be a unital quantale. Then* $\mathrm{R}(Q)$ *is initial in* ${}_Q\overline{\mathbf{Mod}}$, *and* $\mathrm{L}(Q)$ *is initial in* $\overline{\mathbf{Mod}}_Q$.

Quotients of modules can be studied in the same way as quotients of quantales. For instance, a *nucleus* on a left Q-module L is a closure operator $j : L \to L$ such that $a \cdot j(x) \leqslant j(a \cdot x)$ for all $a \in Q$ and $x \in L$. Given a set $R \subseteq L$ there is a least nucleus on L such that $j(y) = j(z)$ for all $(y,z) \in R$, which is denoted by j_R.

We conclude this section on modules by providing a presentation of tensor quantales that generalizes Example 2.6 and which will be used later on.

Definition 2.8. Let Q be a unital quantale and L a unital left Q-module. The *tensor quantale* $\mathscr{T}(Q,L)$ is presented as $\mathbf{Qu}_e\langle Q \amalg L \mid R\rangle$, where the relations in R are the following:

(U) $e = [e_Q]$,

(m) $[a] \cdot [b] = [a \cdot b] \quad (a,b \in Q)$,

(aL) $[a] \cdot [l] = [a \cdot l] \quad (a \in Q,\ l \in L)$,

(VQ) $\displaystyle\bigvee_i [a_i] = \left[\bigvee_i a_i\right] \quad (a_i \in Q)$,

(VL) $\displaystyle\bigvee_i [l_i] = \left[\bigvee_i l_i\right] \quad (l_i \in L)$.

The above terminology is justified because $\mathscr{T}(Q,L)$ is isomorphic to the quantale $\bigoplus_{n\in\omega} L^{\otimes n} \otimes Q$, whose multiplication is defined by, for $n,m \in \mathbb{N}$ (we omit the obvious associativity isomorphisms):

$$(L^{\otimes n} \otimes Q) \otimes (L^{\otimes(1+m)} \otimes Q) \xrightarrow{1_L^{\otimes n} \otimes \alpha \otimes 1_L^{\otimes m} \otimes 1_Q} L^{\otimes(n+1+m)} \otimes Q$$

$$(L^{\otimes n} \otimes Q) \otimes Q \xrightarrow{1_L^{\otimes n} \otimes \mu} L^{\otimes n} \otimes Q,$$

where we write $\mu : Q \otimes Q \to Q$ and $\alpha : Q \otimes L \to L$ for the sup-lattice homomorphisms that define the multiplication of Q and the action of Q on L, respectively (i.e., $\mu(a \otimes b) = a \cdot b$ and $\alpha(a \otimes l) = a \cdot l$). The injection of generators corresponds to the homomorphism $Q \amalg L \to \bigoplus_{n\in\omega} L^{\otimes n} \otimes Q$ defined by $a \mapsto a$ for $a \in Q$, and $l \mapsto l \otimes e_Q$ for $l \in L$.

2.6. Tropological systems

Similarly to ring modules, we define a category $\overline{\mathbf{QMod}}$ of left modules over variable quantale from the contravariant functor $\mathbf{Qu}^{\mathrm{op}} \to \mathbf{CAT}$ that assigns to each unital quantale Q the category ${}_Q\overline{\mathbf{Mod}}$. Its objects are the pairs (Q,L) with L a pre-unital left Q-module, and a morphism $(Q,L) \to (Q',L')$ is a pair (h,k) where $h:Q \to Q'$ is a pre-unital homomorphism of quantales and $k:L \to L'$ is a strong homomorphism of sup-lattices satisfying $k(a\cdot x)=h(a)\cdot k(x)$. It follows that the notion of tropological system, which was already motivated in Section 1.1, is essentially a morphism in $\overline{\mathbf{QMod}}$ targeted at a module $(\mathscr{P}(P\times P),\mathscr{P}(P))$.

Definition 2.9. A *tropological system* $(P,Q,\bar{\to},L,\Pi)$ consists of a set P (of *states*), a unital quantale Q (of *finite observations*), a pre-unital left Q-module L (of *capabilities*), and a morphism $(\bar{\to},\Pi):(Q,L) \to (\mathscr{P}(P\times P),\mathscr{P}(P))$ in $\overline{\mathbf{QMod}}$. For each $a\in Q$, the binary relation $\xrightarrow{a} \subseteq P\times P$ is the *transition relation* of a. We write $p \models \varphi$ for $p\in\Pi(\varphi)$, and define the *behavioural preorder* $\precsim$ on P by $p \precsim q$ if and only if $p\models\varphi \Rightarrow q\models\varphi$ for all $\varphi\in L$. Two states p and q are *behaviourally equivalent*, and we write $p\sim q$, if both $p\precsim q$ and $q\precsim p$. The system $(P,Q,\bar{\to},L,\Pi)$ is said to be *stable* if both $\bar{\to}$ and L are unital, and *unstable* otherwise.

As was already mentioned in Section 1.1, $(P,Q,\bar{\to},L,\Pi)$ is a stable system if and only if the following conditions hold:

(i) $p\xrightarrow{e}q$ if and only if $p=q$,

(ii) $p\xrightarrow{a\cdot b}q$ if and only if $p\xrightarrow{a}r\xrightarrow{b}q$ for some $r\in P$,

(iii) $p\xrightarrow{\bigvee X}q$ if and only if $p\xrightarrow{a}q$ for some $a\in X$,

(iv) $p\models\top_L$,

(v) $p\models\bigvee Y$ if and only if $p\models\varphi$ for some $\varphi\in Y$,

(vi) $p\models a\cdot\varphi$ if and only if $p\xrightarrow{a}q$ and $q\models\varphi$ for some $q\in P$.

By allowing systems to be unstable we bring the theory closer to that of [19], and from the computational point of view we provide a way of dealing with systems capable of performing internal activity hidden from the observer [26,28]. More precisely, this means allowing L to be pre-unital and replacing the first of the above six conditions by the weaker $p\xrightarrow{e}p$ for all $p\in P$ (i.e., making $\bar{\to}$ pre-unital). Notice however that in typical situations L is a quotient of $\mathrm{R}(Q)$ and is thus automatically unital, which means that hidden behaviour is often about making $\bar{\to}$ pre-unital and nothing else.

From the point of view of tropological systems there are also notions of "spatiality", which originally [1] were motivated by the need to characterize the complete presentations of quantales and modules, where completeness is understood in a logical sense and means that from a presentation of a pair (Q,L) by generators and relations it is possible to derive equationally all the "formulas" that are satisfied by all the "models", which are labelled transition systems but in fact can be identified with tropological systems [29,30] (a logic of this kind is presented explicitly in [27,28]). In [1] there is a corresponding notion of "second completeness", for Q, and "third completeness", for L, which however can be formulated independently of generators and relations, and strictly in terms of tropological systems [29,30]. We give here the corresponding definitions.

Definition 2.10. (Q, L) is *second complete* if for all $a, b \in Q$ the equality $a = b$ holds if for all tropological systems $(\bar{\rightarrow}, \Pi) : (Q, L) \to (\mathscr{P}(P \times P), \mathscr{P}(P))$ we have $\xrightarrow{a} = \xrightarrow{b}$. And (Q, L) is *third complete* if for all $\varphi, \psi \in L$ the equality $\varphi = \psi$ holds if for all tropological systems $(\bar{\rightarrow}, \Pi) : (Q, L) \to (\mathscr{P}(P \times P), \mathscr{P}(P))$ we have $\Pi(\varphi) = \Pi(\psi)$. If (Q, L) is both second and third complete we say that it is *complete*.

We have actually given definitions of completeness which differ slightly from those of [1,29] because in the latter all systems were stable. Of course, this can be fixed by allowing, in the above definition, $(\bar{\rightarrow}, \Pi)$ to range only over stable systems.

3. Tropological models, systems, and points

In the present paper we shall adopt the following definition of point, which similarly to [24,22,14] requires representations to be irreducible but not necessarily algebraically irreducible. Nevertheless we use pre-unital homomorphisms in order to bring the theory closer to that of [19].

Definition 3.1. A *point* of a unital quantale Q is a pre-unital irreducible representation of Q. A representation $r : Q \to \mathscr{Q}(\mathscr{P}(P))$ is said to be a *relational representation* on P, and if it is pre-unital and irreducible it is a *relational point*. Points which are unital representations are called *unital points*.

Similarly, for involutive quantales we define involutive points as follows.

Definition 3.2. An *involutive point* of an involutive unital quantale Q is an involutive pre-unital irreducible representation of Q on a self-dual sup-lattice. Involutive relational points and involutive unital points are as in Definition 3.1.

3.1. Presenting quantales in $\overline{\mathbf{Qu}}_e$

We will work with presentations of quantales by generators and relations in $\overline{\mathbf{Qu}}_e$, which can be easily derived from presentations in $\mathbf{Qu}_e$, as follows.

Proposition 3.3. *The forgetful functor U from $\overline{\mathbf{Qu}}_e$ to $\mathbf{Qu}_e$ has a left adjoint that assigns to each unital quantale Q the quantale $Q^\top$ presented by generators and relations as follows, where $t \notin Q$:*

$$Q^\top \stackrel{\mathrm{def}}{=} \mathbf{Qu}_e\langle Q \cup \{t\} \mid [a] \leqslant [t] \quad (a \in Q),$$

$$[t] \cdot [t] \leqslant [t],$$

$$[a \cdot b] = [a] \cdot [b] \quad (a, b \in Q),$$

$$\left[\bigvee_i a_i\right] = \bigvee_i [a_i] \quad (a_i \in Q),$$

$$[e_Q] = e\rangle.$$

Proof. Let $h: Q \to Q'$ be a unital homomorphism into a unital quantale Q'. Let also $h_t: Q \cup \{t\} \to Q'$ be its extension to a map such that $t \mapsto \top_{Q'}$. Clearly, h_t respects all the defining relations of $Q^\top$, and thus it has an extension $\bar{h}: Q^\top \to Q'$ which is a unital homomorphism. Furthermore, $t \mapsto \top_Q$ implies that $\bar{h}$ is strong because $[t]$ is the top of $Q^\top$. Besides, $\bar{h}$ is the only unital strong homomorphism $Q^\top \to Q'$ such that $\bar{h}([a]) = h(a)$ for all $a \in Q$ because being strong defines $\bar{h}([t]) = \top_{Q'}$, and this defines $\bar{h}$ on all the generators of $Q^\top$. Hence, $Q^\top$ has the required universal property, i.e., we have a natural bijection $\mathbf{Qu}_e(Q, UQ') \cong \overline{\mathbf{Qu}}_e(Q^\top, Q')$. $\square$

Corollary 3.4. *Let G be a set, t a symbol not in G, and $R \subseteq (G \cup \{t\})^* \times (G \cup \{t\})^*$. The quantale presented by G and R in $\overline{\mathbf{Qu}}_e$ is*

$$\overline{\mathbf{Qu}}_e\langle G \mid R\rangle \stackrel{\text{def}}{=} \mathbf{Qu}_e\langle G \cup \{t\} \mid R \cup \{[x] \leqslant [t] \mid x \in G\} \cup \{[t] \cdot [t] \leqslant [t]\}\rangle.$$

In applications we shall invariably write $\top$ instead of $[t]$.

3.2. Tropological models

As stated in the introduction, we will work with tropological systems which are also equipped with a right module (of "preparations"), which motivates the following definitions:

Definition 3.5. A *tropological triple* (R, Q, L), or simply a *triple*, consists of a unital quantale Q, a pre-unital right Q-module R, and a pre-unital left Q-module L. The triple (R, Q, L) is *unital* if both R and L are unital modules. Given two triples (R, Q, L) and (R', Q', L'), a *tropological map* $(\rho, \omega, \lambda): (R, Q, L) \to (R', Q', L')$ consists of a pre-unital homomorphism $\omega: Q \to Q'$ (which turns R' and L' into pre-unital Q-modules), a strong homomorphism of right Q-modules $\rho: R \to R'$, and a strong homomorphism of left Q-modules $\lambda: L \to L'$. A map (ρ, ω, λ) is *unital* if ω is a unital homomorphism. The category whose objects are the tropological triples and whose morphisms are the tropological maps is denoted by **TR**.

Given a triple (R, Q, L), the notion of tropological system can be generalized in the obvious way as consisting of a tropological map $(R, Q, L) \to (\mathscr{P}(P), \mathscr{P}(P \times P), \mathscr{P}(P))$, where as before P is the set of states, but now there is also a strong right Q-module homomorphism $\rho: R \to \mathscr{P}(P)$ that assigns to each $r \in R$ the set of states which may have been *prepared* by means of the "preparation procedure" r.

Definition 3.6. Let (R,Q,L) be a tropological triple. A *tropological system* for (R,Q,L) is a tropological map

$$(\rho,\omega,\lambda):(R,Q,L)\to(\mathscr{P}(P),\mathscr{P}(P\times P),\mathscr{P}(P)),$$

where P is the set of *states*. The system is said to be *stable* if both (R,Q,L) and (ρ,ω,λ) are unital, and *unstable* otherwise.

However, the results in this section will apply to a much more general class of tropological maps. Recall that $\mathscr{P}(P)$ is, as a right $\mathscr{P}(P\times P)$-module, isomorphic to $\mathrm{L}(\mathscr{P}(P\times P))$, and that as a left $\mathscr{P}(P\times P)$-module it is isomorphic to $\mathrm{R}(\mathscr{P}(P\times P))$. Hence, tropological systems are essentially examples of tropological maps $(\rho,\omega,\lambda):(R,Q,L)\to(\mathrm{L}(M),M,\mathrm{R}(M))$, where M is a unital quantale. Since the inclusions of $\mathrm{L}(M)$ and $\mathrm{R}(M)$ into M are strong M-module homomorphisms (and thus also Q-module homomorphisms), ρ determines by composition with the inclusion $\mathrm{L}(M)\to M$ a strong Q-module homomorphism $R\to M$ whose image is contained in $\mathrm{L}(M)$, and any such homomorphism arises in the same way from a strong Q-module homomorphism $\rho:R\to\mathrm{L}(M)$. Similar remarks apply to λ, and thus a tropological map $(R,Q,L)\to(\mathrm{L}(M),M,\mathrm{R}(M))$ is uniquely determined by a map m from the disjoint union $R\amalg Q\amalg L$ to M such that m is a "model" in the following sense.

Definition 3.7. Let (R,Q,L) be a tropological triple, and let M be a unital quantale. A *tropological model of* (R,Q,L) *in* M, or simply a *model*, is a map $m:R\amalg Q\amalg L\to M$ whose restrictions to R, Q, and L, satisfy:
(i) $m|_Q$ is a homomorphism of pre-unital quantales;
(ii) $m|_R$ is a strong homomorphism of right Q-modules;
(iii) $m|_L$ is a strong homomorphism of left Q-modules;
(iv) $m(r)$ is left-sided in M for all $r\in R$;
(v) $m(l)$ is right-sided in M for all $l\in L$.

The above five conditions can be conveniently encoded into the following presentation of a quantale by generators and relations.

Definition 3.8. Let (R,Q,L) be a tropological triple. The *spectrum* of (R,Q,L) is the unital quantale $\mathrm{S}(R,Q,L)$ whose presentation by generators and relations in $\overline{\mathbf{Qu}_e}$ has the disjoint union $R\amalg Q\amalg L$ as set of generators, and the following defining relations:

(TR) $\top=[\top_R]$,

(TL) $\top=[\top_L]$,

(TQ) $[\top_Q]\leqslant\top$,

(U) $e\leqslant[e_Q]$,

(m) $[a]\cdot[b]=[a\cdot b]\quad(a,b\in Q)$, (aL) $[a]\cdot[l]=[a\cdot l]\quad (a\in Q,\ l\in L)$,

(aR) $[r]\cdot[a]=[r\cdot a]\quad (r\in R,\ a\in Q)$,

(rs) $[l]\cdot\top\leqslant[l]\quad (l\in L)$,

(ls) $\top\cdot[r]\leqslant[r]\quad (r\in R)$,

(VQ) $\bigvee_i[a_i]=\left[\bigvee_i a_i\right]\quad (a_i\in Q)$,

(VL) $\bigvee_i[l_i]=\left[\bigvee_i l_i\right]\quad (l_i\in L)$,

(VR) $\bigvee_i[r_i]=\left[\bigvee_i r_i\right]\quad (r_i\in R)$.

Given a unital quantale M, a map $m:R\amalg Q\amalg L\to M$ is a model if and only if it respects the above defining relations. The injection of generators $[-]$ is a universal model in the sense that any other model $m:R\amalg Q\amalg L\to M$ factors uniquely through $[-]$ and a strong and unital homomorphism of quantales $\bar{m}:\ \mathrm{S}(R,Q,L)\to M$.

Theorem 3.9. *Let the functor* $\mathrm{Tr}:\overline{\mathbf{Qu}}_e\to\mathbf{TR}$ *be defined as follows*:

- $\mathrm{Tr}(Q)=(\mathrm{L}(Q),Q,\mathrm{R}(Q))$ *for each unital quantale* Q;
- $\mathrm{Tr}(h)=(h\,|_{\mathrm{L}(Q)},h,h\,|_{\mathrm{R}(Q)})$ *for each strong and unital quantale homomorphism* h.

Then Tr *has a left adjoint that to each tropological triple* (R,Q,L) *assigns its spectrum* $\mathrm{S}(R,Q,L)$.

Proof. Tr is clearly a functor, so let (R,Q,L) be a triple, M a unital quantale, and $(\rho,\omega,\lambda):(R,Q,L)\to\mathrm{Tr}(M)$ a tropological map. Let also $m:R\amalg Q\amalg L\to M$ be the model determined by (ρ,ω,λ). From the discussion above it is clear that, given any strong and unital quantale homomorphism $h:\ \mathrm{S}(R,Q,S)\to M$, the condition $h\circ[-]=m$ is equivalent to

$$(h|_{\mathrm{L}(\mathrm{S}(R,Q,L))},h,h|_{\mathrm{R}(\mathrm{S}(R,Q,L))})\circ([-]|_R,[-]|_Q,[-]|_L)=(\rho,\omega,\lambda). \tag{1}$$

Hence, the homomorphic extension $\bar{m}:\ \mathrm{S}(R,Q,L)\to M$ is the unique strong and unital quantale homomorphism that satisfies (1), and thus the triple

$$([-]|_R,[-]|_Q,[-]|_L)$$

provides the unit of the adjunction. □

Corollary 3.10. *Let* (R,Q,L) *be a tropological triple. The tropological systems for* (R,Q,L) *are in bijective correspondence with the relational unital points of* $\mathrm{S}(R,Q,L)$.

Also, from the initiality of $\mathrm{L}(Q)$ as a right Q-module in $\overline{\mathbf{Mod}}_Q$ it follows that $\overline{\mathbf{QMod}}$ is isomorphic to the full subcategory of **TR** whose objects are the triples of the form $(\mathrm{L}(Q), Q, L)$, and we obtain the following equivalence between tropological systems over pairs (Q, L) (i.e., according to Definition 2.9) and points.

Corollary 3.11. *Let Q be a unital quantale and L a pre-unital left Q-module. The tropological systems for (Q, L) are in bijective correspondence with the relational unital points of* $\mathrm{S}(\mathrm{L}(Q), Q, L)$.

Of course, a simpler presentation can be given for $\mathrm{S}(\mathrm{L}(Q), Q, L)$, as follows.

Proposition 3.12. *Let Q be a unital quantale and L a pre-unital left Q-module. Then* $\mathrm{S}(\mathrm{L}(Q), Q, L)$ *is isomorphic to the unital quantale* $\mathrm{S}(Q, L)$ *presented as* $\overline{\mathbf{Qu}}_e\langle Q \amalg L \mid R\rangle$, *where R consists of the following relations*:

(TL) $\top = [\top_L]$,
(TQ) $[\top_Q] \leqslant \top$,
(U) $e \leqslant [e_Q]$,
(m) $[a] \cdot [b] = [a \cdot b] \quad (a, b \in Q)$,
(aL) $[a] \cdot [l] = [a \cdot l] \quad (a \in Q,\ l \in L)$,
(rs) $[l] \cdot \top \leqslant [l] \quad (l \in L)$,
(VQ) $\bigvee_i [a_i] = \left[\bigvee_i a_i\right] \quad (a_i \in Q)$,
(VL) $\bigvee_i [l_i] = \left[\bigvee_i l_i\right] \quad (l_i \in L)$.

Proof. Let i be the inclusion $Q \amalg L \to \mathrm{L}(Q) \amalg Q \amalg L$. It is easy to see that due to the initiality of $\mathrm{L}(Q)$ the map $[-] \circ i : Q \amalg L \to \mathrm{S}(\mathrm{L}(Q), Q, L)$ has the same universal property as the injection of generators $Q \amalg L \to \mathrm{S}(Q, L)$. $\square$

Even simpler presentations can be given in those applications, such as most of the examples in [1,29], where L is a quotient of $\mathrm{R}(Q)$:

Proposition 3.13. *Let Q be a unital quantale, $R \subseteq \mathrm{R}(Q) \times \mathrm{R}(Q)$ a set, and L the left module quotient* $(\mathrm{R}(Q))_{j_R}$ *of* $\mathrm{R}(Q)$ *(see Section 2.5). Then* $\mathrm{S}(Q, L)$ *is isomorphic as a unital quantale to* $\overline{\mathbf{Qu}}_e\langle Q \mid R'\rangle$, *where R' contains the following relations*:

(U) $e \leqslant [e_Q]$,
(m) $[a] \cdot [b] = [a \cdot b] \quad (a, b \in Q)$,
(VQ) $\bigvee_i [a_i] = \left[\bigvee_i a_i\right] \quad (a_i \in Q)$,
(T) $[a] \cdot \top = [b] \cdot \top \quad ((a \cdot \top_Q, b \cdot \top_Q) \in R)$.

Proof. First, the projection functor $\overline{\mathbf{QMod}} \to \mathbf{Qu}$ has a left adjoint F that sends each unital quantale Q' to $(Q', \mathrm{R}(Q'))$ and each pre-unital homomorphism $h: Q' \to Q''$ to (h, h') where $h'(a \cdot \top_{Q'}) = h(a) \cdot \top_{Q''}$ for all $a \in Q'$. The maps $f: Q \to Q'$ that respect the relations in R' are precisely the pre-unital quantale homomorphisms with the property that $f(a) \cdot \top_{Q'} = f(b) \cdot \top_{Q'}$ for all $(a \cdot \top_Q, b \cdot \top_Q) \in R$. Equivalently, these are the maps such that $F(f): (Q, \mathrm{R}(Q)) \to (Q', \mathrm{R}(Q'))$ factors (necessarily uniquely) through $(1_Q, j_R): (Q, \mathrm{R}(Q)) \to (Q, L)$. This suffices to conclude that the map $[-] \circ i: Q \to \mathrm{S}(Q, L)$, where i is the inclusion $Q \to Q \amalg L$ and $[-]$ is the injection of generators of $\mathrm{S}(Q, L)$, has the same universal property as the injection of generators of $\overline{\mathbf{Qu}}_e\langle Q \mid R'\rangle$. $\square$

Example 3.14. Let A be a set, $R \subseteq \mathscr{P}(A^*) \times \mathscr{P}(A^*)$, and $Q = \mathbf{Qu}_e\langle A \mid R\rangle$. The left Q-module $\mathrm{R}(Q)$ is the largest (left module) quotient of Q that identifies e_Q and $\top_Q$ [29, Proposition 3.6], and thus any module quotient of $\mathrm{R}(Q)$ is also a quotient of $\mathscr{P}(A^*)$. Hence, any pair (Q, L) with L a quotient of $\mathrm{R}(Q)$ as in Proposition 3.13 can be described by a tuple (A, R, R'), where both R and R' are subsets of $\mathscr{P}(A^*) \times \mathscr{P}(A^*)$. A stable tropological system for (Q, L) with set of states P is then the same as a transition system labelled over A, such that for all $(a, b) \in R$ we have $p \xrightarrow{a} q \Leftrightarrow p \xrightarrow{b} q$ for all $p, q \in P$, and for all $(a, b) \in R'$ we have $p \xrightarrow{a} \Leftrightarrow p \xrightarrow{b}$ for all $p \in P$ (with $p \xrightarrow{a}$ meaning that $p \xrightarrow{a} q$ for some $q \in P$), where the transition structure $\xrightarrow{-}$ has been freely extended to $\mathscr{P}(A^*)$ in $\mathbf{Qu}_e$. This is at the basis of the "observational logic" of [27,28], whose formulas are defining relations for quantales and modules, and whose models are labelled transition systems as just described. Using Proposition 3.13 an alternative logic can be defined, with only one type of formula, provided that we use an extra symbol $t \notin A$ (the "top"), which denotes a "chaotic" observation that should always be interpreted as the total relation $P \times P$; the formulas are then pairs $(a, b) \in \mathscr{P}((A \cup \{t\})^*) \times \mathscr{P}((A \cup \{t\})^*)$, with obvious adaptations to the case of unital homomorphisms (see Section 3.3 below), and each module relation (a, b) of the original logic is translated to $(a \cdot \{t\}, b \cdot \{t\})$. The two logics are to some extent equivalent, although from a semantically complete theory of the original logic one does not necessarily obtain, using this translation, a complete theory in the new logic. This problem is related to the relationship between completeness of tropological systems and spatiality of quantales, which will be discussed in Section 4.3.

A difference between [19] and the work depicted above is that we are using quantales that are not involutive. However, there is an obvious reformulation of the above results in the setting of involutive quantales. Basically, if we had replaced $\overline{\mathbf{Qu}}_e$ by $\overline{\mathbf{Qu}}_e^*$ in Definition 3.8 we would have obtained a presentation of a unital involutive quantale $\mathrm{S}^*(R, Q, L)$ whose injection of generators $[-]$ is a universal model of the triple (R, Q, L) in unital *involutive* quantales; that is, each model $R \amalg Q \amalg L \to M$, with M involutive, factors uniquely through $[-]$ and a strong and unital involution preserving homomorphism of quantales $\bar{m}: \mathrm{S}_e^*(R, Q, L) \to M$. The new version of Theorem 3.9 would therefore give an adjunction between $\mathbf{TR}$ and $\overline{\mathbf{Qu}}_e^*$, and tropological systems would be identified with those points of the "involutive spectrum" $\mathrm{S}^*(R, Q, L)$ which are unital, relational, and involutive.

The spectrum of a C*-algebra, as defined in [19], is an involutive quantale, but it can be presented by generators and relations without paying any attention to the involution, as indeed the authors remark. More precisely, the result we obtain if we present the spectrum of a unital C*-algebra in $\mathbf{Qu}_e$ (without the axioms for involution) is the same as that which is obtained by presenting the spectrum in $\mathbf{Qu}_e^*$ and additionally requiring the involution of the algebra to be preserved in the presentation. Hence, to some extent involution can be ignored, and it is not wrong to think of the presentations of [19] as if they were in $\mathbf{Qu}_e$.

In Section 4.1 we will further address involutive quantales, in particular discussing an example of how a tropological triple (R, Q, L) can be endowed with an "involution" that can be preserved when presenting $\mathrm{S}^*(R, Q, L)$.

3.3. Pre-unital versus unital homomorphisms

Now we compare what was done in the previous section with the approach outlined in [19] for studying the spectrum of a C*-algebra, namely as regards the role of unital and pre-unital homomorphisms. In doing so we define a notion of unital spectrum $\mathrm{S}_e(R, Q, L)$ of a tropological triple (R, Q, L), we show that the unital spectrum allows us to characterize the stable tropological systems for a unital triple (R, Q, L) as being the unital relational points of $\mathrm{S}_e(R, Q, L)$, and we extend this characterization to the localic tropological systems of [30].

The points of [19] are pre-unital homomorphisms, not necessarily unital. However, if we see the theory of the spectrum of a unital C*-algebra described in [19] as being a presentation in $\mathbf{Qu}_e$, with the formula *true* playing the role of our unit e, the models of a C*-algebra A in a unital quantale Q correspond precisely to *unital* homomorphisms from the "Lindenbaum algebra" $\mathrm{Lind}\,A$ of the theory, which is the quantale presented, to Q. In order to understand the reason behind pre-unital homomorphisms, we begin by remarking that there is a functor $\mathrm{Un} : \mathbf{Qu} \to \mathbf{Qu}_e$ which is left adjoint to the inclusion functor $\mathbf{Qu}_e \to \mathbf{Qu}$.

Proposition 3.15. $\mathbf{Qu}_e$ *is a reflective subcategory of* $\mathbf{Qu}$.

Proof. The unit of the adjunction is, for each pre-unital quantale Q, the injection of generators $Q \to \mathbf{Qu}_e\langle Q \mid R\rangle$, where R consists of the following defining relations:

$$e \leqslant [e_Q],$$

$$[a] \cdot [b] = [a \cdot b] \quad (a, b \in Q),$$

$$\bigvee_i [a_i] = \left[\bigvee_i a_i\right] \quad (a_i \in Q). \qquad \square$$

From the proof of [19, Theorem 3.1] one concludes that $\mathrm{Lind}\,A$ is isomorphic to $\mathrm{Un}(\mathrm{Max}\,A)$, where $\mathrm{Max}\,A$ is the quantale of closed subspaces of A. Hence, unital homomorphisms from Lind A can be identified with pre-unital homomorphisms from Max A.[1]

A similar situation can occur in the case of tropological models. Let the *unital spectrum* $\mathrm{S}_e(R,Q,L)$ of a triple (R,Q,L) be defined exactly like $\mathrm{S}(R,Q,L)$ but with the defining relation $e \leqslant [e_Q]$ replaced by $e = [e_Q]$. Then any unital tropological map $(\rho,\omega,\lambda):(R,Q,L) \to (\mathrm{L}(M),M,\mathrm{R}(M))$ can be identified with a strong and unital homomorphism $\mathrm{S}_e(R,Q,L) \to M$ via an adjunction similar to that of Theorem 3.9. In the case that (R,Q,L) is a unital triple (i.e., R and L are unital Q-modules), we can identify tropological models with strong and pre-unital homomorphisms from $\mathrm{S}_e(R,Q,L)$, as we now show.

Theorem 3.16. *If* (R,Q,L) *is a unital triple*, $\mathrm{S}(R,Q,L)$ *and* $\mathrm{Un}(\mathrm{S}_e(R,Q,L))$ *are isomorphic as unital quantales.*

Proof. First of all it is easy to verify that if M is a unital quantale and $\tau \in M$ satisfies $\tau \geqslant e$ and $\tau \cdot \tau = \tau$ then the set $\tau M \tau = \{\tau \cdot a \cdot \tau \mid a \in M\}$ defines a subquantale of M, which has unit τ and therefore the inclusion $\tau M \tau \to M$ is a pre-unital homomorphism. The modules of left- and right-sided elements are given by $\mathrm{L}(\tau M \tau) = (\mathrm{L}(M))\tau = \{l \cdot \tau \mid l \in \mathrm{L}(M)\}$ and $\mathrm{R}(\tau M \tau) = \tau(\mathrm{R}(M)) = \{\tau \cdot r \mid r \in \mathrm{R}(M)\}$. If $(\rho,\omega,\lambda):(R,Q,L) \to (\mathrm{L}(M),M,\mathrm{R}(M))$ is a tropological map then ω factors through a unital quantale homomorphism $Q \to \tau M \tau$ and the inclusion $\tau M \tau \to M$, where $\tau = \omega(e_Q)$. If R is unital, $\rho: R \to \mathrm{L}(M)$ factors through the inclusion $(\mathrm{L}(M))\tau \to \mathrm{L}(M)$, which is a homomorphism of right Q-modules, because being unital tells us that any $r \in R$ satisfies $r \cdot e = r$ and thus we have $\rho(r) \cdot \tau = \rho(r) \cdot \omega(e_Q) = \rho(r \cdot e_Q) = \rho(r)$, i.e., $\rho(r) \in (\mathrm{L}(M))\tau$. Similar remarks apply to L, and we conclude that for a unital triple (R,Q,L) any tropological map $(R,Q,L) \to (\mathrm{L}(M),M,\mathrm{R}(M))$ factors uniquely through the unital map $(R,Q,L) \to (\mathrm{L}(\tau M\tau),\tau M\tau,\mathrm{R}(\tau M\tau))$. Such a map determines a unital model $m: R \amalg Q \amalg L \to \tau M \tau$, and m extends uniquely to a strong and unital quantale homomorphism $\bar{m}: \mathrm{S}_e(R,Q,L) \to \tau M \tau$, which can be identified with a strong and pre-unital homomorphism $\mathrm{S}_e(R,Q,L) \to M$ such that the unit is mapped to τ. Now notice that the adjunction between $\mathbf{Qu}$ and $\mathbf{Qu}_e$ whose left adjoint is Un restricts to an adjunction between $\overline{\mathbf{Qu}}$ and $\overline{\mathbf{Qu}}_e$ because for any unital quantale Q' the injection of generators $Q' \to \mathrm{Un}(Q')$ is strong (it suffices to see that the new unit e satisfies $e \vee [\top_{Q'}] \leqslant [e_{Q'} \vee \top_{Q'}] = [\top_{Q'}]$) and thus the preceding discussion shows that a tropological model $R \amalg Q \amalg L \to M$ extends uniquely, when R and L are unital, to a strong and unital quantale homomorphism $\mathrm{Un}(\mathrm{S}_e(R,Q,L)) \to M$; that is, the unital quantale $\mathrm{Un}(\mathrm{S}_e(R,Q,L))$ has the same universal property as the spectrum $\mathrm{S}(R,Q,L)$. □

[1] In fact in [19, Theorem 3.1] the statement is that Max A is isomorphic to Lind A, which is a consequence of interpreting the theory of the spectrum of a C*-algebra in $\mathbf{Qu}$ instead of $\mathbf{Qu}_e$. However, notice that it is not possible in general to present quantales by generators and relations in $\mathbf{Qu}$ because unless one of the generators is interpreted as the unit of the quantale being presented the uniqueness of the homomorphic extensions of the assignments to the generators, which is required by a universal property, is lost, and so we prefer to work with $\mathbf{Qu}_e$ instead of $\mathbf{Qu}$.

Hence, to some extent in the case of unital triples, $\mathrm{S}_e(R,Q,L)$ plays the role that $\operatorname{Max} A$ plays for a unital C*-algebra A, and moreover it enables us to characterize stable tropological systems.

Corollary 3.17. *If L is a unital left Q-module, the tropological systems for (Q,L) can be identified with the pre-unital relational points of* $\mathrm{S}_e(\mathrm{L}(Q),Q,L)$, *and the stable tropological systems correspond to the unital relational points.*

From now on we denote by $\mathrm{S}_e(Q,L)$ the unital quantale which is presented like $\mathrm{S}(Q,L)$ (see Proposition 3.12) but with the defining relation $e \leqslant [e_Q]$ replaced by $e=[e_Q]$. Of course, $\mathrm{S}_e(\mathrm{L}(Q),Q,L)$ and $\mathrm{S}_e(Q,L)$ are isomorphic as unital quantales, and thus $\mathrm{S}_e(Q,L)$ provides a simpler presentation of $\mathrm{S}_e(\mathrm{L}(Q),Q,L)$.

The unital spectrum of a pair (Q,L) can also be applied to the notion of localic tropological system which was put forward in [30] with the aim of establishing some results of [1] in a constructive form. The main difference is that the sets of states are replaced by locales, and the powerset left modules $\mathscr{P}(P)$ are replaced by frames ΩP.[2] Given a unital quantale Q and a unital left Q-module L, such systems are defined to consist of a locale P such that ΩP is a unital left Q-module, equipped with a strong left Q-module homomorphism $\Pi : L \to \Omega P$, and satisfying $a\cdot x = a\cdot\top_{\Omega P}\wedge x$ for all $a\in Q$ such that $a \leqslant e$ and all $x\in \Omega P$, a property which is referred to as the *stability axiom*. Hence, a system is (at least if we ignore some of the constructive issues behind [30]) the same as a unital tropological map $(\mathrm{L}(Q),Q,L) \to (\mathrm{L}(\mathscr{Q}(\Omega P^{\mathrm{op}})),\mathscr{Q}(\Omega P^{\mathrm{op}}),\mathrm{R}(\mathscr{Q}(\Omega P^{\mathrm{op}})))$, and thus corresponds to a unital point of $\mathrm{S}_e(Q,L)$ targeted at a (sup-lattice) endomorphism quantale of a coframe, or, equivalently, a left $\mathrm{S}_e(Q,L)$-module structure on ΩP which is both unital and irreducible. Furthermore, we can characterize those unital points which arise from systems, as follows.

Lemma 3.18. *Let Q be a unital quantale, L a unital left Q-module, and $a\in \mathrm{S}_e(Q,L)$. Then $a\leqslant e$ if and only if $a=[b]$ for some $b\in Q$ such that $b\leqslant e_Q$.*

Proof. In the unital spectrum we have $e=[e_Q]$, whence $[a]\leqslant e$ for all $a\in Q$ such that $a\leqslant e_Q$. For the converse direction we sketch a specific construction of $\mathrm{S}_e(Q,L)$. First, we note that $\mathrm{S}_e(Q,L)$ is a quotient of the tensor quantale $\mathscr{T}(Q,L)$ of Definition 2.8, which can be concretely described as consisting of those subsets $I\subseteq L^*\times Q$ such that $(s(\bigvee X)t,a)\in I \Leftrightarrow \{(slt,a)\mid l\in X\}\subseteq I$ and $(s,\bigvee Y)\in I \Leftrightarrow \{(s,a)\mid a\in Y\}\subseteq I$ for all $s,t\in L^*$, $l\in L$, $X\subseteq L$, $a\in Q$, and $Y\subseteq Q$, with multiplication $I\cdot J$ being the least such set that contains the pointwise multiplication of I and J, where the multiplication on elements of $L^*\times Q$ is given by, for all $s,t\in L^*$ and $a,b\in Q$,

$$(s,a)\cdot(t,b)=\begin{cases}(s(a\cdot l)u,b) & \text{if } t=lu \text{ for some } l\in L \text{ and } u\in L^*,\\ (s,a\cdot b) & \text{if } t=\varepsilon\end{cases}$$

and its unit is (ε,e_Q) (see Definition 2.8 and the comments after it). Then $\mathrm{S}_e(Q,L)$ is the quantale quotient of the tensor quantale by the quantic nucleus generated by

[2] If P is a locale, we adopt the convention of writing ΩP when we think of P as an object of the algebraic category of frames rather than its dual category of "localic spaces".

the two axioms (rs) and $[\top_Q] \leqslant [\top_L]$, and thus $\mathrm{S}_e(Q,L)$ can be identified with the set of those $I \subseteq L^* \times Q$ that satisfy the above closure properties and also the following additional ones:

$$(x_1 \dots x_n \top_L y_1 \dots y_m, a) \in I \Rightarrow (x_1 \dots x_n \top_Q y_1 \dots y_m, a) \in I$$
$$(x_1 \dots x_n z \top_L y_1 \dots y_m, a) \in I \Leftrightarrow (x_1 \dots x_n z y_1 \dots y_m, a) \in I \quad (z \in L).$$

Let us call such closed sets S_e-*ideals*. For all $a \in Q$ the set $[a] = \{(\varepsilon, b) \mid b \leqslant a\}$ is clearly an S_e-ideal (and thus $[a]$ is the image of a given by the injection of generators), and in particular $[e_Q]$ is the unit of $\mathrm{S}_e(Q,L)$. Also, $[e_Q]$ does not contain any pair (s,a) with $s \neq \varepsilon$, and thus the subunits (i.e., elements below the unit) of $\mathrm{S}_e(Q,L)$ correspond precisely to the principal ideals of Q which are generated by subunits of Q. □

Theorem 3.19. *Let P be a locale, whose frame of opens is denoted by ΩP. The localic tropological systems for (Q,L) with locale of states P are in bijective correspondence with the unital points $\mathrm{S}_e(Q,L) \to \mathscr{Q}(\Omega P^{\mathrm{op}})$ for which the corresponding left action of $\mathrm{S}_e(Q,L)$ on ΩP satisfies $a \cdot x = a \cdot \top_{\Omega P} \wedge x$ for all $x \in \Omega P$ and all $a \in \mathrm{S}_e(Q,L)$ such that $a \leqslant e$.*

Proof. By the previous lemma, the condition $a \cdot x = a \cdot \top_{\Omega P} \wedge x$ for $a \leqslant e$ in $\mathrm{S}_e(Q,L)$ is equivalent to the stability axiom. □

4. Factor models

In this section we study models of tropological triples in factor quantales, where a *factor quantale*, or simply a *factor*, is a quantale whose only two-sided elements are 0 and $\top$ [22]. Such factor models cover many cases that occur in practice, for any quantale of sup-lattice endomorphisms $\mathscr{Q}(S)$ is a factor [22], and so in particular are the Hilbert quantales of [20], of which $\mathscr{P}(P \times P)$ is a special case. In fact unital factors also include a much more general class of quantales, namely the quantales of endomorphisms of strong and dense Galois connections between sup-lattices [25], of which examples are the quantales of continuous endomaps of dense closure spaces that have been studied in the context of quantum physics [4,7].

Factors are important in the characterization of simple quantales [22], and in this section we see that they also play an important role in connection with tropological models. More precisely, we will see that in the case of models in factor quantales (i) the adjunction of Theorem 3.9 preserves enough structure so as to avoid loss of information about observable behaviour when moving from (R,Q,L) to $\mathrm{S}(R,Q,L)$; (ii) a natural notion of map of models of (R,Q,L), which generalizes maps of tropological systems (see Section 4.2), can be characterized in terms of $\mathrm{S}(R,Q,L)$, leading to an isomorphism of categories between the category of models for a given triple and a suitable category of "modules" of its spectrum; and (iii) we study, albeit preliminarily, the extent to which completeness of a triple (R,Q,L) is related to spatiality of its spectrum.

4.1. Behaviour preservation

Let (R,Q,L) be a tropological triple, M a unital quantale, and $(\rho,\omega,\lambda):(R,Q,L)\to(\mathrm{L}(M),M,\mathrm{R}(M))$ a tropological map. We shall call an element $\rho(r)\in\mathrm{L}(M)$ a *concrete preparation*, and an element $\lambda(l)\in\mathrm{R}(M)$ is a *concrete capability*. The concrete preparations form a submodule $\rho[R]$ of $\mathrm{L}(M)$, and the concrete capabilites form a submodule $\lambda[L]$ of $\mathrm{R}(M)$. In the case where M is $\mathscr{P}(P\times P)$ for some set P, the set of concrete capabilities determines the behaviour preorder $\lesssim\subseteq P\times P$ (see Section 1.1), for we have

$$p\lesssim q \quad\Leftrightarrow\quad \forall_{\varphi\in L}(p\in\lambda(\varphi)\Rightarrow q\in\lambda(\varphi)).$$

In this section we are concerned with those situations in which the translation of systems to points does not cause loss of information about the behaviour preorder. For this it suffices to show that the lattice of concrete capabilities can be recovered from the homomorphism $\bar{m}:\ \mathrm{S}(R,Q,L)\to M$ which is obtained from (ρ,ω,λ) by the adjunction of Theorem 3.9. Of course, this is trivial if we consider points of $\mathrm{S}(R,Q,L)$ without forgetting the unit of the adjunction $(R,Q,L)\to(\mathrm{L}(M),M,\mathrm{R}(M))$, but we would like to be able to characterize systems solely in terms of the spectrum of a triple, and thus we want to recover the concrete capabilities from $\bar{m}$ alone. In this section we will see that this is possible if M is a unital factor. More precisely, we will see that if M is a unital factor then the lattices of concrete preparations and concrete capabilities satisfy

$$\rho[R]=\bar{m}[\mathrm{L}(\mathrm{S}(R,Q,L))],$$
$$\lambda[L]=\bar{m}[\mathrm{R}(\mathrm{S}(R,Q,L))].$$

We start by establishing the following property of unital factors, whose proof in a more general case is due to Kruml.

Proposition 4.1. *Let Q be a unital factor quantale. Then,*

$$x\cdot x'=\begin{cases} x & \textit{if } x'\neq 0,\\ 0 & \textit{otherwise,}\end{cases}\qquad y'\cdot y=\begin{cases} y & \textit{if } y'\neq 0,\\ 0 & \textit{otherwise,}\end{cases}$$

for all $x,x'\in\mathrm{R}(Q)$ and $y,y'\in\mathrm{L}(Q)$.

Proof. We check right-sided elements first. Let $x,x'\in\mathrm{R}(Q)$. If $x'=0$ then of course $x\cdot x'=0$, so let us assume $x'\neq 0$ and prove that $x\cdot x'=x$. First, we remark that $\top\cdot x'$ is two-sided, and thus $\top\cdot x'\in\{0,\top\}$. Since Q is unital, $\top\cdot x'\geqslant e\cdot x'=x'\neq 0$, and thus $\top\cdot x'=\top$. Hence, $x\cdot x'=(x\cdot\top)\cdot x'=x\cdot(\top\cdot x')=x\cdot\top=x$. The proof for left-sided elements is similar. □

Theorem 4.2. *Let (R,Q,L) be a tropological triple and M a unital factor quantale. Let also $(\rho,\omega,\lambda):(R,Q,L)\to(\mathrm{L}(M),M,\mathrm{R}(M))$ be a tropological map, and let $\bar{m}:\ \mathrm{S}(R,Q,L)\to M$ be the strong and unital quantale homomorphism determined by (ρ,ω,λ). Then,*

$$\rho[R]=\bar{m}[\mathrm{L}(\mathrm{S}(R,Q,L))],\tag{2}$$

$$\lambda[L]=\bar{m}[\mathrm{R}(\mathrm{S}(R,Q,L))].\tag{3}$$

Proof. $\mathrm{S}(R,Q,L)$ is a quotient of $\mathscr{P}((R \amalg Q \amalg L)^*)$ and thus each one of its elements is the image of a set of sequences from $(R \amalg Q \amalg L)^*$. For the remainder of this proof we shall write only $\top$ instead of $\top_R$ or $\top_L$, and in order to simplify notation we identify the sequences $s \in (R \amalg Q \amalg L)^*$ with their images in $\mathrm{S}(R,Q,L)$. For each $s \in (R \amalg Q \amalg L)^*$, the sequence $s\top$ denotes a right-sided element of $\mathrm{S}(R,Q,L)$, and any right-sided element is a join of these. Similarly, any left-sided element is a join of elements of the form $\top s$. From here on we focus on right-sided elements. Our aim is to prove that for each sequence $s\top$ there exists $l \in L$ such that $\bar{m}(s\top) = \lambda(l)$. For the empty sequence $s = \varepsilon$ we let $l = \top_L$. Proceeding by induction on the length of s, assume that $\bar{m}(s\top) = \lambda(l)$, and consider three cases, where $r \in R$, $a \in Q$ and $l \in L$:

- $\bar{m}(rs\top) = \bar{m}(r) \cdot \bar{m}(s\top) = \rho(r) \cdot \lambda(l)$; in any quantale, the multiplication of a left-sided element and a right-sided element (in this order) is two-sided, and thus $\bar{m}(rs\top) = \rho(r) \cdot \lambda(l) \in \{0_M, \top_M\} = \{\lambda(0_L), \lambda(\top_L)\}$;
- $\bar{m}(as\top) = \bar{m}(a) \cdot \bar{m}(s\top) = \omega(a) \cdot \lambda(l) = \lambda(a \cdot l)$;
- $\bar{m}(ls\top) = \bar{m}(l) \cdot \bar{m}(s\top) = \lambda(l) \cdot \lambda(l')$; by Proposition 4.1, $\lambda(l) \cdot \lambda(l') \in \{0_M, \lambda(l)\} = \{\lambda(0_L), \lambda(l)\}$.

Hence, for all sequences s, $\bar{m}(s\top)$ coincides with $\lambda(l)$ for some $l \in L$. This obviously extends to sets of sequences, and thus $\bar{m}[\mathrm{R}(\mathrm{S}(R,Q,L))]$ coincides with $\lambda[L]$. The proof for left-sided elements is analogous. $\square$

It should be noted that in the involutive case this preservation no longer holds. The reason behind this is that R contributes more right-sided elements to $\mathrm{S}_e^*(R,Q,L)$ due to the involution; that is, each $r \in R$ provides $[r]$, a preparation, and its time reversed version $[r]^*$, a capability. Similarly, L adds new left-sided elements. In order to see that this really gives us new concrete properties and capabilities, consider a very simple example of tropological system $(\rho,\omega,\lambda) : (R,Q,L) \to (\mathscr{P}(P), \mathscr{P}(P \times P), \mathscr{P}(P)$ in which $Q = \mathscr{P}(A^*)$ is freely generated by the set A, and $R = \mathrm{L}(Q)$, $L = \mathrm{R}(Q)$. Using the arrow notation described in Section 1.1, we have $p \in \rho(\top \cdot a)$ if and only if $q \xrightarrow{a} p$ for some $q \in P$, and $p \in \lambda(a \cdot \top)$ if and only if $p \xrightarrow{a} q$ for some $q \in P$. For instance, the two black states in the figure below are equivalent in terms of their capabilities (they both can do a and only a), but not in terms of their preparations (the left state could have been prepared by doing a, which is not the case with the right state).

$$\begin{tikzcd} \circ \arrow[d, "a"] & \\ \bullet \arrow[d, "a"] & \bullet \arrow[d, "a"] \\ \circ & \circ \end{tikzcd}$$

Hence, the involutive map $\bar{m} : \mathrm{S}_e^*(R,Q,L) \to \mathscr{P}(P \times P)$ that corresponds to (ρ,ω,λ) has a concrete capability $\bar{m}((\top \cdot a)^*) = \bar{m}(a^* \cdot \top)$ that contains the left black state but not the right one, and this concrete capability does not coincide with one of the original tropological system because the black states are equivalent in terms of their (future) behaviour.

Of course, this happens because we are limiting ourselves to a particular notion of tropological triple which completely ignores the involution. If instead we adopt the point of view that some preparations R correspond to time reversals of capabilities in L, then we can add a defining relation $[r]=[l]^*$ to the presentation of $\mathrm{S}_e^*(R,Q,L)$ whenever r is the time-reversed version of l. If all the preparations are made to arise from capabilities in this way, and if that fact is taken into account in the presentation of $\mathrm{S}_e^*(R,Q,L)$ then it follows that applying involution to them yields back their original form as capabilities, and thus no new concrete capabilities are introduced. For instance, if an "involutive triple" is defined to consist of a triple (R,Q,L) such that Q is involutive, equipped with a binary relation $\theta \subseteq R\times L$, we may add the following defining relations to the presentation of $\mathrm{S}_e^*(R,Q,L)$, for all $a\in Q$, $r\in R$ and $l\in L$ such that $(r,l)\in\theta$:

$$[a]^* = [a^*],$$

$$[r]^* = [l].$$

If θ is total then every $r\in R$ will be such that $[r]^* = [l]$ for some $l\in L$, and thus no new capabilities are introduced.

4.2. Categories of models

There are at least two different notions of morphism of tropological system. One was put forward in [29] with the aim of describing a notion of implementation of systems on other systems. However, the more natural one is that which is described in [30]. In terms of tropological triples, morphisms can be defined as follows.

Definition 4.3. Let the following be tropological systems:

$$(\rho,\omega,\lambda) : (R,Q,L) \to (\mathscr{P}(P),\mathscr{P}(P\times P),\mathscr{P}(P)),$$
$$(\rho',\omega',\lambda') : (R,Q,L) \to (\mathscr{P}(P'),\mathscr{P}(P'\times P'),\mathscr{P}(P')).$$

A *map* from the former to the latter is a homomorphism $f : \mathscr{P}(P) \to \mathscr{P}(P')$ of right Q-modules such that $f\circ\rho = \rho'$ and whose dual left Q-module homomorphism $\hat{f} : \mathscr{P}(P') \to \mathscr{P}(P)$ satisfies $\hat{f}\circ\lambda' = \lambda$.

This definition can be made more general, and in this section we address the very general case in which the quantales of binary relations are replaced by unital factors, and where the morphisms are continuous maps of Galois connections in the sense of [25].

Definition 4.4. Let M and M' be quantales. A *Galois map* from M to M' is a pair (f,g) of sup-lattice homomorphisms $f : \mathrm{L}(M) \to \mathrm{L}(M')$ and $g : \mathrm{R}(M') \to \mathrm{R}(M)$ such that for all $x\in\mathrm{L}(M)$ and $y\in\mathrm{R}(M')$ the following *continuity* condition holds:

$$f(x)\cdot y = 0_{M'} \;\Leftrightarrow\; x\cdot g(y) = 0_M.$$

A Galois map (f,g) is said to be *strong* if f is strong, and *dense* if g is strong.

The terminology "dense" is motivated by the following proposition, where by a *dense* sup-lattice homomorphism h we mean a homomorphism that satisfies $h(x)=0 \Leftrightarrow x=0$, following the notion of dense nucleus in locale theory [12].

Proposition 4.5. *Let M and M' be unital factor quantales, and $(f,g): M \to M'$ a Galois map.*
(i) *If (f,g) is dense then f is dense.*
(ii) *If (f,g) is strong then g is dense.*

Proof. (i) Assume that (f,g) is dense, i.e., that g is strong, and let $x \in \mathrm{L}(M)$. If $f(x)=0$ we have $f(x)\cdot\top_{M'}=0$ which, by continuity, is equivalent to $x\cdot g(\top_{M'})=0$ and thus to $x\cdot\top_M=0$. This, in turn, implies $x=0$ because, since M is unital, $0=x\cdot\top_M \geqslant x\cdot e_M=x$. So we have $f(x)=0 \Rightarrow x=0$, i.e., f is dense.

(ii) Obviously, (g,f) is a Galois map from M' to M, and thus from the previous case we conclude that g is dense if f is strong. □

The following definition provides a generalization of the notion of map of tropological systems.

Definition 4.6. Let $m: R \amalg Q \amalg L \to M$ and $m': R \amalg Q \amalg L \to M'$ be two models of a tropological triple (R,Q,L). By a *map* of models from m to m' we mean a Galois map $(f,g): M \to M'$ such that

(i) f is a homomorphism of right Q-modules,
(ii) g is a homomorphism of left Q-modules,
(iii) $m'|_R = f \circ m|_R$,
(iv) $m|_L = g \circ m'|_L$.

Now we show that maps of models of a tropological triple in factor quantales can be described solely in terms of the spectrum of the triple.

Theorem 4.7. *Let M and M' be unital factor quantales, and let*

$$m: R \amalg Q \amalg L \to M,$$
$$m': R \amalg Q \amalg L \to M',$$

be models of a tropological triple (R,Q,L). Let also $(f,g): M \to M'$ be a Galois map. The following are equivalent:
(i) *(f,g) is a map of models $m \to m'$;*
(ii) *f is a strong homomorphism of right $\mathrm{S}(R,Q,L)$-modules and g is a strong homomorphism of left $\mathrm{S}(R,Q,L)$-modules.*

The $\mathrm{S}(R,Q,L)$-module structures of $\mathrm{L}(M)$, $\mathrm{R}(M)$, $\mathrm{L}(M')$ and $\mathrm{R}(M')$ are those obtained from their M- and M'-module structures via the change of quantale induced by the homomorphic extensions $\bar{m} : \mathrm{S}(R,Q,L) \to M$ and $\bar{m}' : \mathrm{S}(R,Q,L) \to M$, respectively.

Proof. (i) $\Rightarrow$ (ii) Assume $(f,g) : m \to m'$ is a map. Conditions (iii) and (iv) of Definition 4.6 imply that f and g are strong because both m and m' (and thus $m|_R$, etc.) are. In order to see that f and g are $\mathrm{S}(R,Q,L)$-module homomorphisms it suffices to verify that they preserve the action of the generators of $\mathrm{S}(R,Q,L)$, i.e., of arbitrary $r \in R$, $a \in Q$, and $l \in L$. The case $a \in Q$ is immediate because f and g are Q-module homomorphisms. Let then $r \in R$ and $x \in \mathrm{L}(M)$, and let us verify that $f(x \cdot r) = f(x) \cdot r$. In order to make the notation clear we shall distinguish the multiplication from the action by omitting the symbol "$\cdot$" in the case of the action—so the equation we want to verify becomes $f(xr) = f(x)r$. First, $xr = x \cdot \bar{m}(r) = x \cdot m(r)$ by definition. Since both x and $m(r)$ are left-sided elements of M, and M is a unital factor, it follows from Proposition 4.1 that $xr = m(r)$ if $x \neq 0$ and $xr = 0$ if $x = 0$. Hence,

$$f(xr) = \begin{cases} f(m(r)) = m'(r) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Now we compute $f(x)r$, which by definition equals $f(x) \cdot \bar{m}'(r) = f(x) \cdot m'(r)$. Again by the properties of left-sided elements of unital factors we obtain

$$f(x)r = \begin{cases} m'(r) & \text{if } f(x) \neq 0, \\ 0 & \text{if } f(x) = 0. \end{cases}$$

Since g is strong, it follows by Proposition 4.5 that $f(x) = 0 \Leftrightarrow x = 0$, and thus $f(xr) = f(x)r$. Now let $l \in L$ and $x \in \mathrm{L}(M)$. We have $f(x)l = f(x) \cdot m'(l)$, and thus by continuity of (f,g) we have $f(x)l = 0$ if and only if $x \cdot g(m'(l)) = 0$. Since $g \circ m'|_L = m|_L$ and $x \cdot m(l) = xl$ we obtain the equivalence $f(x)l = 0 \Leftrightarrow xl = 0$, which gives us $f(x)l = 0 \Leftrightarrow f(xl) = 0$ because f is dense. Finally, both $f(x)l$ and xl are two-sided elements and thus $f(x)l \in \{0_{M'}, \top_{M'}\}$ and $xl \in \{0_M, \top_M\}$. Since f is strong we conclude $f(x)l = f(xl)$, which ends the proof that f is a homomorphism of $\mathrm{S}(R,Q,L)$-modules. The proof that g is a homomorphism of $\mathrm{S}(R,Q,L)$-modules is entirely analogous.

(ii) $\Rightarrow$ (i) Assume that f is a strong homomorphism of right $\mathrm{S}(R,Q,L)$-modules. Then f is also a Q-module homomorphism via the injection $[-] : Q \to \mathrm{S}(R,Q,L)$, so let us prove $f \circ m|_R = m'|_R$. Let $r \in R$.

$$\begin{aligned} f(m(r)) &= f(\top_M \cdot m(r)) \quad \text{(because } m(r) \text{ is left-sided)} \\ &= f(\top_M r) \\ &= f(\top_M)r \\ &= \top_{M'} r \\ &= \top_{M'} \cdot m'(r) \\ &= m'(r) \quad \text{(because } m'(r) \text{ is left-sided).} \end{aligned}$$

In a similar way one proves that $g \circ m'|_L = m|_L$, and thus (f,g) is a map $m \to m'$. $\square$

Corollary 4.8. *Let the following be tropological systems*:

$$(\rho,\omega,\lambda):(R,Q,L)\to(\mathscr{P}(P),\mathscr{P}(P\times P),\mathscr{P}(P)),$$
$$(\rho',\omega',\lambda'):(R,Q,L)\to(\mathscr{P}(P'),\mathscr{P}(P'\times P'),\mathscr{P}(P')).$$

A function $f:\mathscr{P}(P)\to\mathscr{P}(P')$ *is a map of tropological systems if and only if it is a strong homomorphism of right* $\mathrm{S}(R,Q,L)$*-modules.*

Hence, the category of tropological systems for a triple (R,Q,L) is isomorphic to the category of irreducible relational right $\mathrm{S}(R,Q,L)$-modules (i.e., those modules which are isomorphic to a complete atomic Boolean algebra and whose action determines an irreducible representation) with strong homomorphisms.

4.3. Spatiality and completeness

From the point of view of topology, spatiality of quantales is meant to be a generalization of that of locales, although as we have discussed in Section 2.2 more than one generalization is possible. From the point of view of tropological systems there are analogous notions of completeness, as we have seen in Section 2.6. An issue left to be discussed is therefore the relation there may be between the notions of completeness for tropological triples and the concept (or concepts) or quantale spatiality. We now give a few preliminary remarks about this, in the case of the spatiality of [22,14], which seems to be more directly related to the notion of completeness of Section 2.6, in particular second completeness. In order to compare the two notions we adopt the following definition.

Definition 4.9. Let C be a class of unital quantales. A tropological triple (R,Q,L) is *C-complete* if the following conditions hold:

- for all $x,y\in R$, if for all maps $(\rho,\omega,\lambda):(R,Q,L)\to(\mathrm{L}(M),M,\mathrm{R}(M))$ with $M\in C$ we have $\rho(x)=\rho(y)$, then $x=y$;
- for all $x,y\in Q$, if for all maps $(\rho,\omega,\lambda):(R,Q,L)\to(\mathrm{L}(M),M,\mathrm{R}(M))$ with $M\in C$ we have $\omega(x)=\omega(y)$, then $x=y$;
- for all $x,y\in L$, if for all maps $(\rho,\omega,\lambda):(R,Q,L)\to(\mathrm{L}(M),M,\mathrm{R}(M))$ with $M\in C$ we have $\lambda(x)=\lambda(y)$, then $x=y$.

A unital quantale Q is C-*spatial* if for all $x,y\in Q$, if for all strong and unital quantale homomorphisms $h:Q\to M$ with $M\in C$ we have $h(x)=h(y)$, then $x=y$.

The usual notion of completeness of a triple (R,Q,L) is obtained by specifying the class C to consist of the quantales of binary relations $\mathscr{P}(P\times P)$. The definition of C-spatiality is almost the usual definition of spatiality when C consists of quantales of sup-lattice endomorphisms, except that we are requiring our homomorphisms to be unital.

It is clear that a quantale $\mathrm{S}(R,Q,L)$ is usually not spatial with respect to any class of factor quantales because in order to be so it would have to be *semi-idempotent*, by which we mean a quantale whose right-sided elements and left-sided elements are all idempotent. This is because every unital factor is semi-idempotent (this follows from Proposition 4.1), and because spatiality is defined using strong homomorphisms. So let us define $\mathrm{S}'(R,Q,L)$ to be the largest quantale quotient of $\mathrm{S}(R,Q,L)$ that makes all the right-sided and left-sided elements of $\mathrm{S}(R,Q,L)$ idempotent. Then $\mathrm{S}'(R,Q,L)$ is also the largest semi-idempotent quotient of $\mathrm{S}(R,Q,L)$, by virtue of the following proposition.

Proposition 4.10. *Let Q be a quantale and let*

$$R = \{(a \cdot a, a) \mid a \in \mathrm{L}(Q)\} \cup \{(b \cdot b, b) \mid b \in \mathrm{R}(Q)\}.$$

Then the quotient quantale Q_R is semi-idempotent.

Proof. For any surjective quantale homomorphism $h : Q \to Q'$ and any $a' \in \mathrm{R}(Q')$ there exists $a \in \mathrm{R}(Q)$ such that $h(a) = a'$, where we can take $a = \bigvee\{x \in Q \mid h(x) \leqslant a'\}$ [24]. Hence, any right-sided element of Q' is the image of a right-sided element of Q, and thus it is idempotent. A similar argument applies to left-sided elements. □

In order to give an example of the kind of result that would be interesting to establish we state and prove below a simplified result that addresses only triples of the form $(2,2,L)$, and a restricted form of spatiality whereby we are only concerned with a basis—i.e., a join-dense subset—of $\mathrm{S}'(2,2,L)$, namely the basis consisting of the image of the free monoid L^* in $\mathrm{S}'(2,2,L)$.

Definition 4.11. A class of sup-lattices D is *closed under principal ideals* if for all $S \in D$ and $x \in S$ the principal ideal generated by x is also in D.

For instance, the class of complete and atomic Boolean algebras is closed under principal ideals: for each $X \subseteq P$ the principal ideal generated by X in $\mathscr{P}(P)$ is $\mathscr{P}(X)$. Hence, the theorem below applies in particular to the usual notion of completeness for tropological systems.

From now on we identify each triple $(2,2,L)$ with L, writing, e.g. $\mathrm{S}'(L)$ instead of $\mathrm{S}'(2,2,L)$—notice that this is a quotient of the tensor quantale $\bigoplus_{n\in\omega} L^{\otimes n}$ (see Example 2.6). Accordingly, we say L is *C-complete* if $(2,2,L)$ is, i.e., if there are "enough" strong sup-lattice homomorphisms from L into sup-lattices $\mathrm{L}(Q)$ with Q in the class C. In fact in the following theorem we adopt a notion of *strong C-completeness*, whereby L is *strongly C-complete* if for any finite set of inequalities $x_i \nleqslant y_i$ with $x_i, y_i \in L$ ($i = 1,\ldots,n$) there exists a quantale $Q \in C$ and a strong sup-lattice homomorphism $h : L \to \mathrm{R}(Q)$ such that $h(x_i) \nleqslant h(y_i)$ for all i. The definition of strong completeness generalizes in an obvious way to arbitrary tropological triples, and is not necessarily too strong a requirement; for instance, all the examples of tropological pairs (Q,L) in [1,29] for which completeness was proved were shown to be strongly complete with respect to the class of quantales of binary relations. In fact, even more than that, embeddings $(Q,L) \to (\mathscr{P}(P \times P), \mathscr{P}(P))$ were obtained. As an example of a sufficient condition for strong C-completeness, if C is closed under the formation of finite products then any C-complete triple is strongly C-complete.

From now on we simplify notation by identifying the sequences $x_1 \ldots x_n \in L^*$ with their images $[x_1] \cdot \ldots \cdot [x_n]$ in $\mathrm{S}'(L)$.

Theorem 4.12. *Let D be a class of sup-lattices closed under principal ideals, and let C be the class of all the quantales $\mathcal{Q}(S)$ with $S \in D$. Let also L be a strongly C-complete sup-lattice. Then, for all $s, t \in L^*$, the images of s and t in $\mathrm{S}'(L)$ coincide if and only if for all strong and unital quantale homomorphisms $h : \mathrm{S}'(L) \to \mathcal{Q}(S)$ with $S \in D$ we have $h(s) = h(t)$.*

Proof. The image of L^* in $\mathrm{S}'(L)$ satisfies the following conditions, for all $x, y, z \in L$:

- $x\top = x$,
- $xx = x$,
- $xyz = xzy$.

The last condition is a partial commutativity which holds because the right-sided elements of $\mathrm{S}'(L)$ form an idempotent right-sided quantale [31, Proposition 5.1.1]. The last two conditions allow us to replace each sequence $x_0 \ldots x_n \in L^*$ by the pair $(x_0, \{x_1, \ldots, x_n\})$. The first two conditions are equivalent to the condition $xy = x$ for all $x \leqslant y$, and thus each pair (x, Φ) with $x \in L$ and $\Phi \subseteq_{\mathrm{fin}} L$ can be taken to satisfy $y \nleqslant z$ for all $y, z \in \Phi$ such that $y \neq z$ (i.e., Φ is discrete as a subposet of L), and $x \nleqslant y$ for all $y \in \Phi$; hence, $x = 0 \Rightarrow \Phi = \emptyset$. Since $(x, \{0\})$ represents the multiplication $x0$, we may further restrict to pairs (x, Φ) with $0 \notin \Phi$, in which case we are left with only one pair that represents the bottom of $\mathrm{S}'(L)$, namely $(0, \emptyset)$ [of course, this restriction only eliminates the pairs of the form $(x, \{0\})$, for if $y, z \in \Phi$ with $y \neq z$ we necessarily have $y, z \neq 0$ by the previous restrictions]. If $h : L \to \mathrm{R}(M)$ is a strong sup-lattice homomorphism with M a semi-idempotent quantale we can define a unique unital quantale homomorphism $\bar{h} : \mathrm{S}'(L) \to M$ that extends h; on each pair $(x, \{x_1, \ldots, x_n\})$ it is defined by $(x, \{x_1, \ldots, x_n\}) \overset{\bar{h}}{\mapsto} h(x) \cdot h(x_1) \cdot \ldots \cdot h(x_n)$, with $(x, \emptyset) \overset{\bar{h}}{\mapsto} h(x)$.

Now let (x, Φ) and (y, Ψ) be two such pairs, such that $(x, \Phi) \neq (y, \Psi)$. Then either $x \neq y$ or $\Phi \neq \Psi$. Let us consider the former first, assuming without loss of generality that $x \nleqslant y$. Then $x \neq 0$ and, since also $0 \notin \Phi$, there exists due to strong C-completeness a strong sup-lattice homomorphism $h : L \to \mathrm{R}(\mathcal{Q}(S))$ with $S \in D$ such that $h(x) \nleqslant h(y)$, $h(x) \neq 0$, and $h(x') \neq 0$ for all $x' \in \Phi$. Hence, in $\mathcal{Q}(S)$ we have $\bar{h}(x, \Phi) = h(x) \cdot \prod_{x' \in \Phi} h(x') = h(x)$, by Proposition 4.1, because $h(x)$ and all $h(x')$ are right-sided elements of a unital factor. On the other hand, $\bar{h}(y, \Psi) = h(y) \cdot \prod_{y' \in \Psi} h(y')$ is either $h(y)$ or 0, and in either case this value is different from $h(x)$, which means that the homomorphic extension of h to $\mathrm{S}'(L)$ separates (x, Φ) and (y, Ψ).

Now consider the second possibility, namely $x = y$ and $\Phi \neq \Psi$. Without loss of generality assume $z \in \Phi \setminus \Psi$. Since $\Phi \neq \emptyset$ we must have $x \neq 0$, and thus there exists a sup-lattice $S \in D$ and a strong sup-lattice homomorphism $h : L \to \mathrm{R}(\mathcal{Q}(S))$ such that $h(x) \neq 0$ and, for all $x' \in \Phi \cup \Psi$, also $h(x') \neq 0$ (due to strong C-completeness). Now there are again two possibilities: (i) either $x' \nleqslant z$ for all $x' \in \Psi$ (and also $y \nleqslant z$

because $y = x$), or (ii) there is $x' \in \Psi$ such that $x' \leqslant z$. Recalling that $\mathrm{R}(\mathscr{Q}(S))$ and S^{op} are order-isomorphic we identify h with a strong sup-lattice homomorphism $L \to S^{\mathrm{op}}$. For case (i) let S' be the principal ideal of S generated by $h(z)$, which is also in D. Then S'^{op} is a principal filter of S^{op}, and the map $(- \bigvee h(z)) : S^{\mathrm{op}} \to S'^{\mathrm{op}}$ is a surjective sup-lattice homomorphism that maps $h(z)$ to the zero of S'^{op} but $h(y)$ and all $h(x')$ with $x' \in \Psi$ to a value above zero. Using now the isomorphism $S'^{\mathrm{op}} \cong \mathrm{R}(\mathscr{Q}(S'))$ we obtain a surjection $\mathrm{R}(\mathscr{Q}(S)) \to \mathrm{R}(\mathscr{Q}(S'))$ and, by composition with h, a strong sup-lattice homomorphism $h' : L \to \mathrm{R}(\mathscr{Q}(S'))$ that maps z to zero but y and all $x' \in \Psi$ to a value above zero. Hence, its homomorphic extension $\bar{h}' : \mathrm{S}'(L) \to \mathscr{Q}(S')$ maps (x, Φ) to zero because $z \in \Phi$, but $\bar{h}'(y, \Psi)$ equals $h'(y)$, which is not zero. In case (ii), for which there is $x' \in \Psi$ such that $x' \leqslant z$ (and necessarily $x \nleqslant x'$ and $x'' \nleqslant x'$ for all $x'' \in \Phi$), we use a similar strategy, but constructing another $h' : L \to S'^{\mathrm{op}}$ that annihilates x' but not x or any $x'' \in \Phi$, whose homomorphic extension thus annihilates (y, Ψ) but not (x, Φ). $\square$

A full result for spatiality would state that if (R, Q, L) is C-complete then $\mathrm{S}'(R, Q, L)$ is C-spatial, but the validity of this assertion is unclear. Proving that $\mathrm{S}'(R, Q, L)$ has a "spatial basis"—which itself is not clear for arbitrary triples—is certainly not enough, as for instance the free quantale generated by the basis as a monoid—i.e., the powerset of the basis with multiplication calculated pointwise—is a quantale with the same basis but which is not necessarily semi-idempotent (the powerset of an idempotent monoid is not necessarily an idempotent quantale), and therefore not C-spatial.

Acknowledgements

The author acknowledges the support of the Fundação para a Ciência e Tecnologia through the Research Units Funding Program and through grant POCTI/1999/MAT/33018 "Quantales".

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